The notion of electrical resistance[*]

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A discussion is made of the equations and boundary conditions that govern the flow of electric current in solids under applied voltages, the resistance of a sphere being calculated as an example. The linear relationship between potential difference and electric current across a segment of circuit of arbitrary shape is discussed, and the linear constant —the electrical resistance— is explicity expressed in terms of some effective length and cross-section.

Keywords: electric resistance, electric currents in solids, Laplace's equation and boundary value problems.

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Introduction

Almost two centuries ago the german physicist Georg Simon Ohm (1789-1854) used the electrochemical cell recently invented by Alessandro Volta to experiment with circuits. He discovered then that the potential difference V along a segment of a circuit is proportional to the current intensity I traversing it[1]:

$$V = R\,I. \tag{1}$$

The law that now bears his name introduced for the first time the proportionality constant that we now call electrical resistance R, characteristic of a segment of circuit of which the prototype, though a very specialized one, is a resistor. Ohm verified its experimental validity for right cylinders of circular cross section, but the linear relationship is valid for all sorts of geometries, including regions of (theoretically) unlimited extension.

The validity of eq. (1) depends on the fulfillment of the relationship

$$\vec{J} = \sigma \vec{E}, \tag{2}$$

the local Ohm's Law that relates current density $\vec{J}$ to electric field $\vec{E}$ in a material of electric conductivity σ. This relationship is valid only for solid, isotropic and homogeneous conducting materials in low intensity applied electric fields, in the absence of applied magnetic fields, thermal and stress effects. In liquids the diffusion of inhomogeneous electrolytes may lead to the appearance of internal electromotive forces. In anisotropic materials, like single crystals, $\vec{J}$ and $\vec{E}$ are not colinear and σ is a tensorial magnitude. In inhomogeneous solids, where σ is a function of position, additional effects like contact potentials may come into play. Semiconductors and superconducting materials are excluded because additional considerations are needed. In the presence of applied magnetic fields the Hall Effect must be taken into account, and the application of heat and stress introduces other

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[1] G. S. Ohm, Schweig. J. **46**, 137 (1826).

effects[2]. For high intensity electric fields σ is a nonlinear function of E and the proportionality between V and I does not hold.

Many people consider that resistance is only a dissipative property characteristic of a particular segment of circuit. The dissipative character is universally valid because eq. (2) is a consequence of the fact that —in the low velocity range where drag is proportional to the first power of velocity— the drift velocity of a particle submitted to friction is proportional to the applied force[3]. On the other hand, it is not rigourosly true that resistance is a characteristic of an isolated part of a circuit. As will be shown in detail, resistance is a consequence of the flow of current which stems from an electric field distribution determined by the reciprocal influence of the geometry of the segment of circuit under analysis (the "resistor") and the rest of the circuit. Resistance is a localized property only as refers to current flow, but a property of the whole circuit in respect to the origin and configuration of the electric field. The non-local character is usually hidden in theoretical analysis because in the few soluble cases discussed strong non-explicit assumptions are made about the influence of the rest of the circuit. From the practical point of view the non-local effects are masked because resistors and circuits are —or rather, should be— designed to minimize them.

The calculation of R has been made only for a few geometries[4,5,6,7] apart from the isotropic and homogeneous right cylinders of circular cross section studied by Ohm. An apparently more general expression is obtained when the resistance is related to the capacity of homogeneous solid conductors embedded in dielectrics[8,9]. This approach transfers the burden to

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2 J. F. Nye, *Physical properties of crystals* (Oxford University Press, England, 1979), Fig. 10.1.a.

3 J. R. Reitz, F. J. Milford and R. W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, 1979), pp. 140-154.

4 V. C. Poor, *Electricity and Magnetism* (Wiley, London, 1931), pp. 81-83.

5 J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), pp. 14-15.

6 W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, 1955), p. 108.

7 P. Badoux, *Electricité*, vol. 1 (Presses Académiques Européennes, Brussels, 1959), pp. 54-55.

8 E. Weber, *Electromagnetic Fields* (Wiley, New York, 1950), p. 68.

9 R. W. Wangness, *Electromagnetic Fields* (Wiley, New York, 1979), pp. 237-238.

the calculation of capacities, but has the merit of making obvious that resistance is a property of the whole circuit, a fact well stressed for capacity.

A proof that eq. (1) follows from eq. (2) may be obtained from linearity arguments[10], but the method neither needs nor provides values for R. An explicit general expression of R for bodies of arbitrary shapes could clarify the notion or resistance, but none is found in the best known textbooks on Electromagnetism or in journals devoted to physics education. The importance of the subject certanly merits the thorough discussion that will be given here.

Laws governing current distribution

Under an applied voltage a current distribution is stablished inside a material. After some delay —the duration of the transitory regime— the macroscopic current density acquires a time-independent but space-dependent value, the steady-state regime value $\vec{J}(\vec{r})$. From eq. (2) it follows that current lines and electric field lines coincide. In most electric circuits of practical importance voltage is applied to a piece of conductor by wires of small cross-section. The small contact surface of these wires with the piece of conductor is usually taken to have constant potential, which simplifies calculations but is probably untrue in most cases, as illustrated below for the sphere.

Complete circuits of practical interest are solved using different methods than those discussed below, and will not be considered in this paper. The focus is put here on the resistance of a piece of conductor that is part of a larger electric circuit. The two connections of this piece with the rest of the circuit —that in what follows will be called the electrodes— must be equipotential in order for the potential difference V between them to be well defined. The current I passing through the piece must also be well defined, so there cannot be ramifications, leaks or surface emission of electrons. The electrodes and the external walls must determine a closed surface completely enclosing the conducting material whose resistance is to be determined.

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[10] C. A. Coulson, *Electricity* (Oliver and Boyd, Edinburgh, 1965), p. 79.

The two main causes of the behavior of currents are the large long-range value of Coulomb forces —which forces the electric neutrality of materials— and the mobility of electrons in conductors —which allow them to freely displace. Any macroscopic volume density of charge ς inside the piece of conductor would make free electrons to move away from the regions where it is negative, in such a way that its value inside the material will finally vanish. For the electrostatic case the consequence is that the electric field becomes zero inside and that any net charge is located on the conductor's surface. In the steady-state regime the interior charge density ς vanishes —a fact whose consequence is eq. (11)— but the electric field inside the conductor does not, behaviour that requires explanation.

The two Maxwell´s equations that determine the electric field are[11]

$$\nabla \bullet \vec{E} = \frac{\varsigma}{\varepsilon_0}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3}$$

where ε_0 is the permittivity of vacuum[12] and $\vec{B}$ the magnetic induction. In the steady-state all time derivatives vanish and the only source of electric field is ς. As discussed before, there cannot be charges localized within the material. It is customary to assign fictive charge densities to the electrodes, but this is only a way of replacing the rest of the circuit by assigning values to the normal electric field there, as will be done in the next section. As there are no macroscopic charges inside the material, it follows that $\vec{E}$ is there solenoidal:

$$\nabla \bullet \vec{E} = 0. \tag{4}$$

From here it follows that the discontinuity of the normal component of the electric field across the external surface of the segment of circuit is[13]

$$E_{\perp}^{\text{int}} - E_{\perp}^{\text{ext}} = \frac{\lambda}{\varepsilon_0}. \tag{5}$$

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[11] The International System of Units (SI) is used throughout this paper.

[12] Remember that the treatment is restricted to the isotropic linear range.

[13] Reitz and Milford, op. cit., p. 90.

λ, the surface density of charge on the walls, cannot be determined here because the calculation of the external electric field requires solving the whole circuit.

In dissipative electric circuits the electric field does not vanish inside the conductor because the power supplied by electromotive forces maintains macroscopic net charges on their terminals[14] —and therefore its potential difference— through continuous forced replacement of the moving charges. One may argue that fields are also determined by the charges on the conductor's external surface, which is true, but this is not enough, as is illustrated by the vanishing value obtained in the electrostatic case.

As all magnitudes become time-independent when the steady-state regime is stablished, the second of eqs. (3) becomes

$$\nabla \times \vec{E} = 0, \tag{6}$$

so that the electric field is an irrotational vector. A mathematical property of such vectors is that they may be derived from a scalar potential ϕ. Therefore

$$\vec{E} = -\nabla \phi, \tag{7}$$

$$\nabla \bullet \nabla \phi = \nabla^2 \phi = 0. \tag{8}$$

The minus sign in eq. (7) is chosen so that the forces on positive charges tend to move them from higher to lower potentials, as masses do in gravitational fields. Eq. (8) is known as Laplace’s equation and its solutions are called harmonic functions. The harmonic potential ϕ is determined by solving Laplace’s equation with the boundary conditions that will be given next[15].

From eq. (7) it follows that the potential difference V between any two points $\vec{r}_1$, $\vec{r}_2$ —work done against the field— is always given by

$$V = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \bullet d\vec{l} = \int_{\vec{r}_1}^{\vec{r}_2} \nabla \phi \bullet d\vec{l} = \phi(\vec{r}_2) - \phi(\vec{r}_1), \tag{9}$$

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[14] The terminals might coincide with the electrodes, but it is no necessarily so.

[15] Stratton, op. cit., pp. 222-223.

where $d\vec{l}$ is the vectorial element of line, and the value of the curvilinear integral is independent of the path chosen to go from the first to the second point. This path-independence property is the main characteristic of a field derived from a scalar potential ϕ (conservative field) and will be used below.

An important consequence of eq. (6) is that the tangential component of $\vec{E}$ is continuous across the interface of two materials[16,17],

$$\vec{E}_{\parallel}^{(1)} = \vec{E}_{\parallel}^{(2)}, \tag{10}$$

which explicitly shows that as the electric field does not vanish inside the conductor (case in which there will be no electric current), it does not uniformly vanish outside it.

From eq. (4) it follows that

$$\nabla \bullet \vec{J} = 0. \tag{11}$$

A mathematical consequence of this solenoidal character of $\vec{J}$ is that[18]

$$\oiint_S \vec{J} \bullet d\vec{S} = 0, \tag{12}$$

where *S* is any simply connected closed surface enclosing part of or the whole conductor, and $d\vec{S}$ is the vectorial element of area. The closed surface *S* may be taken to be the one formed by two arbitrary cross-sections of conductor, S_1 and S_2, and the part of wall surface Σ in between them. In that case eq. (12) becomes

$$\oiint_S \vec{J} \bullet d\vec{S} = \iint_{S_1} \vec{J} \bullet d\vec{S} + \iint_{\Sigma} \vec{J} \bullet d\vec{S} + \iint_{S_2} \vec{J} \bullet d\vec{S} = 0. \tag{13}$$

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[16] Reitz and Milford, op. cit., p. 90.

[17] P. Baruch, M. Hulin and J. F. Pétroff, *Electricité et Magnetisme* (Hermann, Paris, 1972), pp. 150-151.

[18] Reitz and Milford, op. cit., p. 13.

As no current flows through the walls —current leaks and thermoionic effects being excluded— in the steady-state regime $\vec{J}$ is tangential there, so that

$$\iint_{\Sigma} \vec{J} \bullet d\vec{S} = 0 \quad \text{and} \quad \iint_{S_1} \vec{J} \bullet d\vec{S} + \iint_{S_2} \vec{J} \bullet d\vec{S} = 0, \tag{14}$$

Current intensity I is defined as the absolute value of the flux of $\vec{J}$ over any cross-section S,

$$I = \left| \iint_{S} \vec{J} \bullet d\vec{S} \right|, \tag{15}$$

and, as a consequence of eq. (14), eq. (15) always gives the same value for any cross-section S.

Because $\vec{J}$ is tangential to the walls, from eq. (2) the interior normal component of the electric field vanishes there,

$$\vec{E}_{\perp} = \vec{E} \bullet \hat{n} = 0, \tag{16}$$

where $\hat{n}$ is the unit vector normal to the wall. This condition, which is not initially valid, as was previously discussed, is obtained by the accumulation of surface charge at the walls during the transitory regime, as given by eq. (5). Eq. (16) does not hold at the electrodes —in most practical cases only a virtual border with the rest of the circuit— where the inverse condition holds: the normal component of $\vec{J}$ must be different from zero and the tangential component of the field should vanish. This is so because the electrodes are by definition equipotential, so the electric field is there normal although not necessarily of constant magnitude. An striking example of this behaviour is the charged conducting disk discussed by Jackson[19].

The two electrodes and the external walls determine a closed surface. A harmonic potential ϕ is completely and uniquely detetermined by the previously given specifications[20]; unfortunately, there is no general procedure

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[19] Jackson, op. cit., pp. 89-93.

[20] Jackson, op. cit. , p. 16.

for determining the potential for such mixed boundary conditions, so ingenuity is required[21]. The following calculation is such an example.

Electrical resistence of a sphere

A good illustration of the difficulties arising in the calculation of resistance would be a case with non-uniform current densities, variable cross-sections and variable lengths of field-line segments. In most such cases the potential can only be expressed in terms of a series expansion in terms of some symmetry adapted basis functions. Comprehension is better and the use of graphics gets simpler when the potential can be expressed in terms of familiar functions. The only case found by the author which fulfills all these conditions is the spherical piece of resistive material discussed next.

A sphere of radius *a* is connected to the rest of the circuit by two cilindrical wires of radius *b* much smaller than *a*:

$$a \gg b\,, \qquad (17)$$

wires that make contact with the sphere at opposite ends of a diameter, as shown in Figure 1.

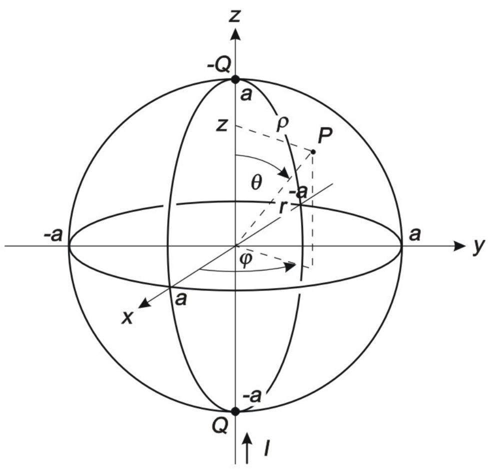

Figure 1. Resistive sphere.

Two boundary conditions determine ϕ. The first one is its constant (and different) value over each of the two electrodes. The second one is that its normal derivative over the rest of the surface of the sphere must vanish.

Due to the wires' small cross-section, in a first approximation the fictive charge distribution on the electrodes that replaces the rest of the circuit may be taken to be two point charges *Q* and –*Q* located on the *z* axis. The coordinates of these charges are respectively -*a* and *a*, chosen so that current *I* flows in the positive sense of the *z* axis (see Figure 1).

[21] Jackson, op. cit., p. 90.

The potential ϕ^{Q} generated at field point $\vec{r}(r,\theta,\varphi)$ by these fictive charges is[22]

$$\phi^{Q}(r,\theta,\varphi) = k_1 Q\left(\frac{1}{d_+(r,\theta)} - \frac{1}{d_-(r,\theta)}\right), \tag{18}$$

where $d_{\pm}$, the distance from field point $\vec{r}$ to charge $\pm Q$, is given by

$$d_{\pm} = \sqrt{\alpha_{\pm}}, \quad \alpha_{\pm} = \rho^2 + (z \pm a)^2 = r^2 + a^2 \pm 2a \cdot r \cdot \cos(\theta). \tag{19}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \rho = \sqrt{x^2 + y^2}, \quad z = \rho \cdot \cos(\theta). \tag{20}$$

The distances from the surface of the sphere to the charges are

$$d_{\pm}(a,\theta) = \sqrt{2}\, a \sqrt{1 \pm \cos(\theta)}. \tag{21}$$

As $z \leq a$, for points along the z axis (r = 0) it follows that

$$d_{\pm}(0,z) = a \pm z. \tag{22}$$

For points on the equatorial plane ($\theta = \pi/2$ or $z = 0$)

$$d_{\pm}(r,\pi/2) = \sqrt{r^2 + a^2} = \sqrt{\rho^2 + a^2} = d_{\pm}(\rho,0). \tag{23}$$

As the system is invariant under any rotation φ around the z axis (azimuthal symmetry), the potential is φ-independent. This condition is only approximately valid because for real circuits the wires carrying I from the exit to the entry point don´t usually preserve the symmetry.

In spherical coordinates the components of the electric field $\vec{E}$ are[23]

$$\begin{gathered}\vec{E} = -\nabla\phi(r,\theta,\varphi) = E_r \hat{r} + E_\theta \hat{\theta} + E_\varphi \hat{\varphi}, \\ \text{where} \quad E_r = -\frac{\partial\phi}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad E_\varphi = -\frac{1}{r\,\mathrm{sen}\theta}\frac{\partial\phi}{\partial\varphi},\end{gathered} \tag{24}$$

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[22] In SI units $k_1 = 1/4\pi\varepsilon_0 = 9\cdot10^9\cdot N\cdot m^2/C^2$.

[23] Stratton, op. cit., p. 52.

and $E_\varphi = 0$ for systems with azimuthal symmetry.

Therefore, the non-vanishing components of the electric field derived from ϕ^Q are

$$\begin{aligned} E_r^Q(r,\theta) &= k_1 Q\left(\frac{r + a\cdot\cos\theta}{d_+^3} - \frac{r - a\cdot\cos\theta}{d_-^3}\right), \\ E_\theta^Q(r,\theta) &= -k_1 Q\cdot a\cdot\sin(\theta)\left(\frac{1}{d_+^3} + \frac{1}{d_-^3}\right). \end{aligned} \tag{25}$$

$\vec{E}^Q$ does not fulfill the boundary condition eq. (16). This may be seen both from the structure of its field lines in Figure 2 (the dashed ones) and from the value of the normal component $E^Q{}_r$ on the surface of the sphere (see eq. (21)):

$$E_r^Q(a,\theta) = \frac{k_1 Q}{2a}\left(\frac{1}{d_+(a,\theta)} - \frac{1}{d_-(a,\theta)}\right). \tag{26}$$

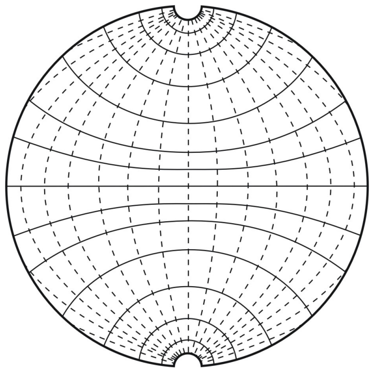

Figure 2. Field and equipotential lines of $\vec{E}^{Q}$[24].

If this were the initial potential, charge would accumulate on the surface of the sphere until $E_r(a,\theta) = 0$. This gives rise to an additional harmonic potential ψ such that

$$E_r^\psi(a,\theta) = -\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = -\frac{k_1 Q}{2a}\left(\frac{1}{d_+(a,\theta)} - \frac{1}{d_-(a,\theta)}\right) = -\frac{1}{2a}\phi^Q(a,\theta) = -E_r^Q(a,\theta) \tag{27}$$

cancels the normal component of $\vec{E}$ on the surface of the sphere. One has to find the potential ψ satisfying eqs. (27) and (8) everywhere inside the sphere, except at the poles.

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[24] In units of $k_1 Q$ the equipotential lines, from bottom to top, correspond to the following values of potential, each half or twice the value of the preceding one, excluding 0: -2.8, -6.4, -3.2, -1.6, -0.8, -0.4, -0.2, 0, 0.2, 0.4, 0.8, 1.6, 3.2, 6.4, 12.8.

Landau and Lifchitz[25] pointed out that if f(r,θ) is a solution of Laplace's equation, so is $\int_0^r \mathrm{f}(r,\theta)\frac{dr}{r}$. One may then take[26]

$$\begin{aligned}\psi(r,\theta) &= \frac{1}{2}\int_0^r \phi^{\mathrm{Q}}(r,\theta)\frac{dr}{r} = \frac{k_1\mathrm{Q}}{2}\int_0^r\left(\frac{1}{d_+(r,\theta)}-\frac{1}{d_-(r,\theta)}\right)\frac{dr}{r}\\ &= \frac{k_1\mathrm{Q}}{2a}\left[\sinh^{-1}\left(\frac{a-r\cos(\theta)}{r\sin(\theta)}\right)-\sinh^{-1}\left(\frac{a+r\cos(\theta)}{r\sin(\theta)}\right)\right]\\ &= \frac{k_1\mathrm{Q}}{2a}\left[\ln\left(\frac{d_-(r,\theta)+a-r\cos(\theta)}{r\sin(\theta)}\right)-\ln\left(\frac{d_+(r,\theta)+a+r\cos(\theta)}{r\sin(\theta)}\right)\right]\\ &= \frac{k_1\mathrm{Q}}{2a}\ln\left(\frac{d_-(r,\theta)+a-r\cos(\theta)}{d_+(r,\theta)+a+r\cos(\theta)}\right),\end{aligned} \tag{28}$$

where ln is the natural logarithm, result that may be verified by derivation. The expression of ψ in terms of sinh^{-1} is more convenient for calculating the components of $\vec{E}$, but is indeterminated at r = 0 where one has to use l'Hôpital´s rule[27] to show it actually vanishes there. The last expression, obtained from a well known identity[28], clearly shows that ψ is finite everywhere except at the poles —where it has a logarithmic divergence— and is, in cylindrical coordinates, an odd function of z.

It may be verified that, barring the poles, ψ satisfies Laplace's equation in spherical coordinates for azimuthal symmetry[29]:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right)=0. \tag{29}$$

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[25] L. Landau and E. Lifchitz, *Electrodynamique des milieux continus* (Éditions Mir, Moscow, 1969), Chapter 3, p. 132. This property, whose proof is sketched in a footnote, is probably restricted to the case where f(0,θ) = 0.

[26] The primitives are given by G. A. Korn and T. M. Korn, *Mathematical Handbook for scientists and engineers* (McGraw-Hill, New York, 1968, 2nd edition), p. 950, eq. 258.

[27] Korn and Korn, op. cit., p. 119, eq. (4.7.8).

[28] M. Abramowitz and I. Stegun, *Handbook of mathematical functions* (Dover, England, 1965, 9th edition), p. 87, eq. (4.6.20).

[29] Stratton, op. cit., p. 52, last of eqs. (95).

The components of the electric field contributed by ψ are

$$E_r^{\psi}(r,\theta)=\frac{k_1Q}{2r}\left[\frac{1}{d_-(r,\theta)}-\frac{1}{d_+(r,\theta)}\right],$$
$$E_\theta^{\psi}(r,\theta)=-\frac{k_1Q}{2ar\sin(\theta)}\left[\frac{r+a\cos(\theta)}{d_+(r,\theta)}+\frac{r-a\cos(\theta)}{d_-(r,\theta)}\right]. \tag{30}$$

Therefore, the total potential ϕ is

$$\phi=\phi^Q+\psi$$
$$=k_1Q\left[\frac{1}{d_+(r,\theta)}-\frac{1}{d_-(r,\theta)}+\frac{1}{2a}\left(\sinh^{-1}\left(\frac{a-r\cos(\theta)}{r\sin(\theta)}\right)-\sinh^{-1}\left(\frac{a+r\cos(\theta)}{r\sin(\theta)}\right)\right)\right] \tag{31}$$
$$=k_1Q\left[\frac{1}{d_+(\rho,z)}-\frac{1}{d_-(\rho,z)}+\frac{1}{2a}\ln\left(\frac{d_-(\rho,z)+a-z}{d_+(\rho,z)+a+z}\right)\right].$$

The field lines of $\vec{E}=\vec{E}^Q+\vec{E}^{\psi}$ are tangential to the surface of the sphere and the equipotential lines are there normal, as shown in Figure 3. The electrodes are taken to be the nearly semi-spherical surfaces of radius b (see Figure 4), the hollows in Figure 3. Although barely noticeable, the "semicircles" here, where $b \cong 0.085\ a$, are slightly larger than those in Figure 2.

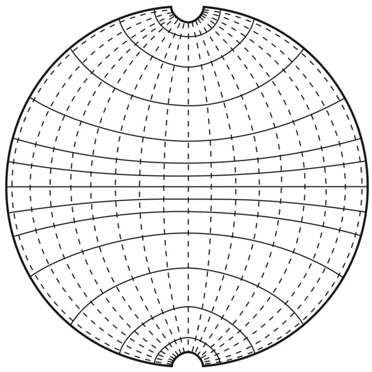

Figure 3. Field and equipotential lines of $\vec{E}=\vec{E}^Q+\vec{E}^{\psi}$[30].

In order to find the resistance R it is necessary to evaluate both the current intensity I given by eq. (15) and the potential difference V between electrodes given by eq. (9).

I is most simply evaluated over an equipotential cross-section, the simplest being the equatorial plane where —being an odd function of z— ϕ vanishes. $\vec{J}$ is there normal to the plane (see Figure 3), its magnitude being

[30] The equipotential lines correspond to the same values as those in Figure 2.

$$\left|\vec{J}(r,\frac{\pi}{2})\right| = -J_\theta(r,\frac{\pi}{2}) = -\sigma \cdot E_\theta(r,\frac{\pi}{2}) = -\sigma\left[E_\theta^Q(r,\frac{\pi}{2}) + E_\theta^\psi(r,\frac{\pi}{2})\right]$$
$$= k_1 Q\sigma\left[a\left(\frac{1}{d_+(r,\frac{\pi}{2})^3} + \frac{1}{d_-(r,\frac{\pi}{2})^3}\right) + \frac{1}{2a}\left(\frac{1}{d_+(r,\frac{\pi}{2})} + \frac{1}{d_-(r,\frac{\pi}{2})}\right)\right] \tag{32}$$
$$= k_1 Q\sigma\left[\frac{2a}{(r^2+a^2)^{\frac{3}{2}}} + \frac{1}{a(r^2+a^2)^{\frac{1}{2}}}\right].$$

From eqs. (15) and (32)

$$I = \left|\iint_\Sigma \vec{J} \bullet d\vec{S}\right| = \iint_\Sigma \left|\vec{J}\right| dS = -\iint_\Sigma J_\theta(r,\frac{\pi}{2}) dS = -\int_0^a 2\pi r J_\theta(r,\frac{\pi}{2}) dr$$
$$= k_1 Q\pi\sigma \int_0^a \left(\frac{2a}{(r^2+a^2)^{\frac{3}{2}}} + \frac{1}{a(r^2+a^2)^{\frac{1}{2}}}\right) d(r^2)$$
$$= k_1 Q\pi\sigma\left[\int_0^{a^2}\left(\frac{2a\,du}{(u+a^2)^{\frac{3}{2}}}\right) + \int_0^{a^2}\frac{du}{a(u+a^2)^{\frac{1}{2}}}\right] \tag{33}$$
$$= k_1 Q\pi\sigma\left[(4-2\sqrt{2}) + (2\sqrt{2}-2)\right] = k_1 Q 2\pi\sigma.$$

The value of the up to now unknown constant Q is now determined, being

$$k_1 Q = \frac{I}{2\pi\sigma}. \tag{34}$$

The potential difference V across the sphere is given by

$$V = \phi(\vec{r}^{\,+}) - \phi(\vec{r}^{\,-}), \tag{35}$$

where $\vec{r}^{\,+}$ is any point on the positive electrode and $\vec{r}^{\,-}$ any point on the negative one. The two electrodes are the end surfaces of the quasi-cylindrical leads through which current I enters and leaves the sphere, the resistance being well defined only if these surfaces are equipotential. As ϕ is an odd function of z,

$$\phi(\rho,-z) = -\phi(\rho,z), \tag{36}$$

a simple choice for V is

$$V = \phi(b,-z_0) - \phi(b,z_0) = 2\phi(b,-z_0). \tag{37}$$

The circle of radius b is the intersection of the external wall of each lead with the surface of the sphere, as shown in Figure 4. The drawing, where the magnitude of b is grossly exagerated, identifies the value z_0 at which the disk of radius b intersects the z axis. It then follows that

$$z_0 = \sqrt{a^2 - b^2} = a\sqrt{1-\left(\frac{b}{a}\right)^2}. \tag{38}$$

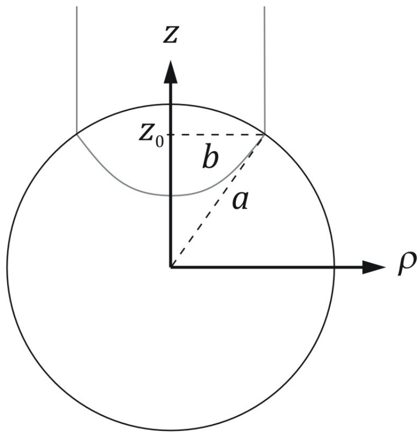

Figure 4. Geometry of electrodes.

Using eqs. (31), (34) and (37) one obtains

$$V = 2\phi(b,z_0) = \frac{I}{\pi\sigma}\left[\frac{1}{d_-(b,z_0)} - \frac{1}{d_+(b,z_0)} + \frac{1}{2a}\ln\left(\frac{d_+(b,z_0)+a+z_0}{d_-(b,z_0)+a-z_0}\right)\right]. \tag{39}$$

The electrical resistance is thererefore given by

$$R = \frac{V}{I} = \frac{1}{\pi\sigma}\left[\frac{1}{d_-(b,z_0)} - \frac{1}{d_+(b,z_0)} + \frac{1}{2a}\ln\left(\frac{d_+(b,z_0)+a+z_0}{d_-(b,z_0)+a-z_0}\right)\right], \tag{40}$$

which becomes infinite when the radius b of the wires goes to zero, and 0 when $b = a$, as expected.

For $b \ll a$ one gets (see eqs. (38) and (19)) to order b/a

$$\begin{gathered}
z_0 = a\sqrt{1-\left(\frac{b}{a}\right)^2} \cong a, \\
d_+(b,z_0) = \sqrt{b^2+(z_0+a)^2} \cong \sqrt{b^2+4a^2} = 2a\sqrt{1+\left(\frac{b}{2a}\right)^2} \cong 2a, \\
d_-(b,z_0) = \sqrt{b^2+(z_0-a)^2} \cong b, \\
\ln\left(\frac{d_-(b,z_0)+a-z_0}{d_+(b,z_0)+a+z_0}\right) \cong \ln\left(\frac{b+a-a}{2a+a+a}\right) = \ln\left(\frac{b}{4a}\right).
\end{gathered} \tag{41}$$

As, from l'Hôpital´s rule,

$$\lim_{x\to 0}\frac{-\ln(x)}{1/x}=\lim_{x\to 0}\frac{-1/x}{-1/x^2}=\lim_{x\to 0}x=0, \tag{42}$$

near the origin 1/*x* is always larger than -ln(*x*), so that the term 1/*b* in *R* predominates over the rest. Figure 5 shows that the relative error made by disregarding the last two terms in eq. (40) is less than 2% for $b/a \le 0{,}01$, and always less than 17% in the whole range $0 < b \le a$.

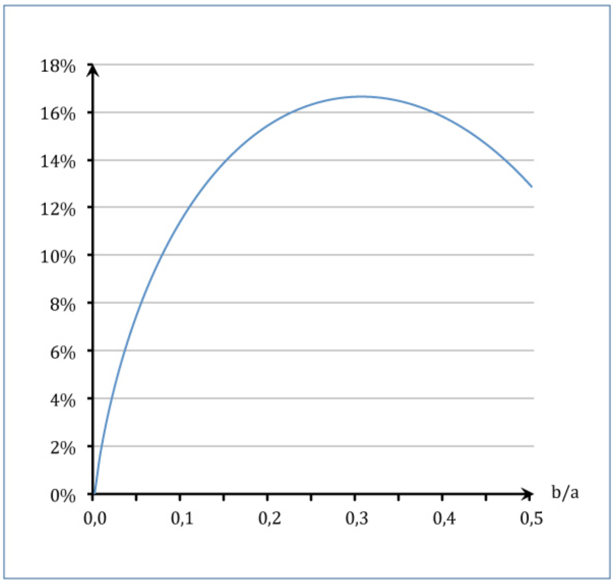

Figure 5. Relative error of the approximate formula eq. (43).

Therefore,

$$R=\frac{V}{I}\cong\frac{1}{\pi\sigma b}\quad\text{when}\quad\frac{b}{a}\ll 1. \tag{43}$$

One should analyze if the given solution reflects general properties of a spherical resistor or is an artifact of the peculiar way in which it was obtained. The main objection that may be raised is the "indentation" of the sphere, the substraccion of a nearly semispherical cap, regardless of its size. One may wonder if a suitable configuration of the external circuit may provide equipotential surfaces that match the spherical surface. Only numerical calculations can answer this, but the author thinks it would not be possible in normal circumstances, barring the use of contact potentials, electrets, extremely high electromotive forces or the use of some kind of special devices. The reason is that current tubes are, as shown by eq. (2), electric field lines of infinitesimal cross section. These tubes tend to be as smooth as possible because only localized large volume charges, can sharply bend them. As equipotential surfaces are normal to current tubes, we cannot expect them to approximate to the shape of the spherical surface without the aforesaid sharp bending. As shown in Figure 6, normal circuit leads would then require the electric field to have a sharp discontinuity on the line where they meet the surface, discontinuity that can only be produced by a localized charge surface which is naturally produced only in sharp edges.

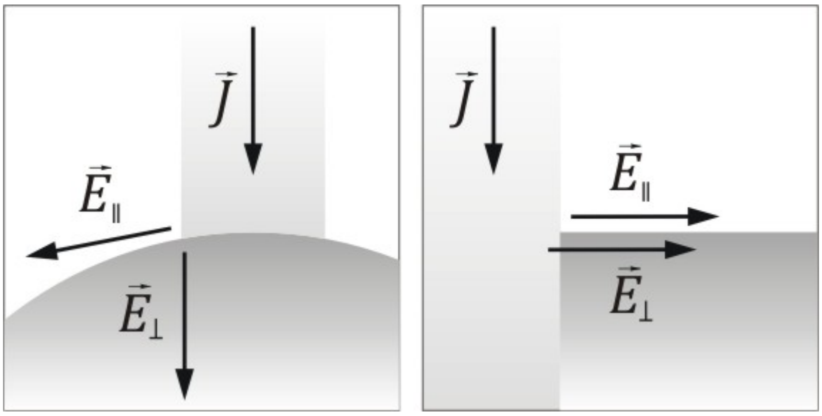

Figure 6. Normal versus tangential field: the problem of electrodes.

Linear relationship of *V* and *I*

Field lines are the natural paths for the calculation of V, as equipotential surfaces are the natural surfaces for the calculation of I. If one uses a coordinate system based on them, all calculations are greatly simplified. The equations that define such a set of coordinates are well known[31] though the functions relating them to the standard cartesian ones would in general be non-elementary trascendental functions. For the previously discussed case of the sphere, for instance, such a set is defined by the surfaces

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \text{constant}, \quad \phi = \text{constant}, \tag{44}$$

[31] See, for instance, N. Kemmer, *Vector analysis. A physicist´s guide to the mathematics of fields in three dimensions* (Cambridge University Press, England, 1977).

and the surface generated by the rotation around the z axis of the field lines defined by the differential equation[32]

$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta}. \tag{45}$$

Such a set of coordinates may be found for any body whose shape can be described by a mathematical function, but not after the problem is fully solved. It would be cumbersome to use if the functions involved are unfamiliar, but the analysis made with it would be as sound as that made in the usual cartesian coordinates. The purpose of this section is not to evaluate the actual value of a resistance in terms of such a coordinate system but, assuming its existence, to find an explicit integral expression of the constant that linearly relates *V* and *I* for any well defined piece of circuit.

A set of orthogonal curvilinear coordinates u_1, u_2, u_3 is defined such that the surfaces u_1 = constant (in what follows, the u_1-surfaces) are equipotential. $\phi(u_1)$ is then independent of u_2 and u_3, but not $\vec{E}$ and $\vec{J}$ as may be seen from eqs. (48). The remaining curvilinear coordinates are defined so that the field lines are the intersection of the u_2 and u_3-surfaces and both are orthogonal to the u_1-surfaces.

The differential elements ds_1 of length along a field line, da_1 of area on an equipotential surface and dv of volume are then given by[33]

$$\begin{aligned} ds_1 &= h_1 du_1, \\ da_1 &= h_2 h_3 du_2 du_3, \\ dv &= h_1 h_2 h_3 du_1 du_2 du_3 \end{aligned} \tag{46}$$

where

$$h_j(u_1,u_2,u_3) = \sqrt{\left(\frac{\partial x_1}{\partial u_j}\right)^2 + \left(\frac{\partial x_2}{\partial u_j}\right)^2 + \left(\frac{\partial x_3}{\partial u_j}\right)^2},$$

and $x_j(u_1,u_2,u_3)$ gives the standard cartesian coordinates in terms of the curvilinear ones.

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[32] Kemmer, op. cit., p. 42. This surface of revolution is a consequence of the azimuthal symmetry.

[33] Stratton, op. cit., pp. 38-50.

For the spherical coordinates r, φ, θ (see Figure 1) these functions are

$$x = r\sin(\theta)\cos(\varphi), \quad y = r\sin(\theta)\sin(\varphi), \quad z = \cos(\theta),$$
$$h_r = 1, \quad h_\varphi = r\sin(\theta), \quad h_\theta = r. \tag{47}$$

the well known factors giving the usual differential lengths dr, $r\sin(\theta)\, d\varphi$ and $r\, d\theta$.

As ϕ is a function only of u_1, eq. (7) may be written in differential form as

$$E = -\frac{d\phi}{ds_1} = -\frac{1}{h_1}\frac{d\phi}{du_1}, \quad \text{or} \quad -d\phi = E \cdot h_1 du_1. \tag{48}$$

If (see Figure 7)

$$\phi_0 = \phi(u_1^{(0)}), \quad \phi_f = \phi(u_1^{(f)}), \tag{49}$$

and

$$\phi_0 > \phi_f \tag{51}$$

the potential difference V is given by

$$V = (\phi_0 - \phi_f) = \int_{\phi_f}^{\phi_0} d\phi = \int_{u_1^{(0)}}^{u_1^{(f)}} E \cdot h_1 du_1. \tag{52}$$

Current I is given by

$$I = \iint_S J\, da_3 = \iint_S J h_2 h_3\, du_2\, du_3, \tag{53}$$

where S is an equipotential cross-section.

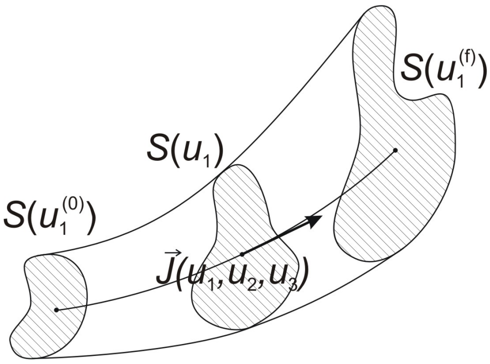

Figure 7. Segment of circuit showing the two electrodes, an intermediate cross-section and a single field line.

Notice that, by definition of the coordinate system, $\vec{J}$ is normal to S, so that its normal component there equals its magnitude, the sign following from the choice $\phi_0 > \phi_f$:

$$|\vec{J}_\perp| = |\vec{J}| = J. \tag{54}$$

In order to find both V and I it is necessary to succesively integrate over the three variables, in a way that resembles a volume integral.

This suggests analyzing the following expression:

$$\Omega = \iiint J\, h_1 h_2 h_3\, du_1\, du_2\, du_3. \tag{55}$$

Ω would yield Ohm's law if the integral could be evaluated in the following two different orders:

$$\Omega = \left(\iint h_2 h_3\, du_2\, du_3\right)\left(\int \sigma E\, h_1\, du_1\right) = \left(\int h_1\, du_1\right)\left(\iint J\, h_2 h_3\, du_2\, du_3\right). \tag{56}$$

The right factors would then yield σV in the first member and I in the second one. The left factors would then be, respectively, the cross-section area A and the lenght L of the enclosed field line segment. For this to be true very stringent physical and geometrical conditions must be met. The piece of conductor must have equipotential cross-sections of constant area; all field line segments must have the same length; h_1 must be a function only of u_1; h_2 and h_3 must both be independent of u_1. Otherwise, integrations cannot be done as indicated.

The previous geometrical conditions are met for right cylinders of circular cross-section A and length L. It then follows that, in cylindrical coordinates,

$$u_1 = z,\ u_2 = \rho,\ u_3 = \varphi,\ h_1 = 1,\ h_2 = 1,\ h_3 = \rho. \tag{57}$$

The physical conditions are met if all cross-sections are equipotential and the electric field inside the cylinder is independent of ρ, φ. These are non-trivial conditions. As mentioned before, φ independence is never rigorously obtained because in real circuits the return circuit for I usually breaks azimuthal symmetry. $\vec{E}$ is uniform inside the conductor if its normal value at the electrodes (the base and top of the cylinder) is uniform. As discussed at the beginning, the equipotential character of the cross-sections forces the tangential component of $\vec{E}$ to vanish there but imposes no condition on its magnitude (the normal component of eq. (54)). The effect is minimized in resistors because the cross-section of its leads is much smaller than that of the body, but the field distribution near its ends is certanly not uniform and the equipotential surfaces there are certanly not plane. The difference in the resulting value of R is not important in normal circuits because the values of resistance have in most cases a tolerance greater than 1%. On the other hand, the effect may be important in microcircuits, where designers have to carefully analyze this problem.

Under these conditions, and taking into account eq. (46), for a cylinder eq. (56) gives

$$\begin{aligned}\Omega &= \left(\iint h_2 h_3\, du_2\, du_3\right)\left(\int \sigma \cdot E \cdot h_1\, du_1\right) = A \cdot \sigma \cdot V \\ &= \left(\int h_1\, du_1\right)\left(\iint J \cdot h_2 h_3\, du_2\, du_3\right) = L \cdot I,\end{aligned} \tag{58}$$

so that

$$R = \frac{V}{I} = \frac{L}{\sigma \cdot A}. \tag{59}$$

For the previously discussed reasons, the value eq. (59) —invariably quoted in Electricity textbooks—, is not an exact one, at most a very good approximation. In the knowledge of the author the right cylinder is the only case where eq. (56) may be used to obtain an explicit expression for *R*. But —as we shall immediately see— integral Ω may nevertheless be used to explicity prove the validity of Ohm's Law. To this end use will be made of the first mean value theorem for integration that follows[34]:

If f(x) and g(x) are continuous functions of x and g(x) ≠ 0 for $x_1 \le x \le x_2$,

$$\int_{x_1}^{x_2} \mathrm{f}(x)\mathrm{g}(x)\,dx = \mathrm{f}(c)\int_{x_1}^{x_2} \mathrm{g}(x)\,dx, \quad \text{where} \quad x_1 \le c \le x_2. \tag{60}$$

Eq. (60) is used twice to obtain *I* from eq. (55), taking into account that from eq. (14) it follows that the surface integral is independent of u_1:

$$\begin{aligned}\Omega &= \iiint J \cdot h_1 h_2 h_3\, du_1\, du_2\, du_3 \\ &= \int_{u_1^{(0)}}^{u_1^{(f)}} \left(\iint\limits_{S(u_1)} h_1(u_1,u_2,u_3) h_2(u_1,u_2,u_3) h_3(u_1,u_2,u_3) J(u_1,u_2,u_3)\, du_2\, du_3 \right) du_1 \\ &= \int_{u_1^{(0)}}^{u_1^{(f)}} h_1(u_1,c_2,c_3) \left(\iint\limits_{S(u_1)} h_2(u_1,u_2,u_3) h_3(u_1,u_2,u_3) J(u_1,u_2,u_3)\, du_2\, du_3 \right) du_1 \\ &= I \int_{u_1^{(0)}}^{u_1^{(f)}} h_1(u_1,c_2,c_3)\, du_1 = L(c_2,c_3) I, \\ &\text{where} \quad u_j^{\min} \le c_j \le u_j^{\max} \quad \text{for} \quad j=2,\ 3.\end{aligned} \tag{61}$$

[34] Korn and Korn, op. cit., p. 119, eq. (4.7.5).

$S(u_1)$ is the equipotential cross section at u_1 (see Figure 7) and u_j^{min}, u_j^{max} are the minimum and maximum values of u_j on $S(u_1)$. $L(c_2,c_3)$ is the length of the segment of flux line $u_2 = c_2$, $u_3 = c_3$ comprised between the equipotential surfaces $S(u_1^{(0)})$ and $S(u_1^{(f)})$.

In a similar fashion integral Ω may be evaluated for V giving

$$\begin{aligned}
\Omega &= \iiint J \cdot h_1 h_2 h_3 \, du_1 \, du_2 \, du_3 = \sigma \iint \left(\int_{u_1^{(0)}}^{u_1^{(f)}} h_1 h_2 h_3 \, E \, du_1 \right) du_2 \, du_3 \\
&= \sigma \iint \left(\int_{u_1^{(0)}}^{u_1^{(f)}} h_1 h_2 h_3 \frac{1}{h_1} \frac{\partial \phi(u_1)}{\partial u_1} du_1 \right) du_2 \, du_3 \\
&= \sigma \left(\iint h_2(c_1,u_2,u_3) h_3(c_1,u_2,u_3) \, du_2 \, du_3 \right) \left(\int_{\phi^{(0)}}^{\phi^{(f)}} d\phi \right) \\
&= \sigma V A(c_1), \quad \text{where} \quad u_1^{(0)} \le c_1 \le u_1^{(f)}.
\end{aligned} \tag{62}$$

$A(c_1)$ is the area of the equipotential cross-section at $u_1 = c_1$.

It then follows that
$$R = \frac{V}{I} = \frac{L(c_2,c_3)}{\sigma A(c_1)}. \tag{63}$$

The formula is similar to the simple one for the right cylinder of circular cross-section, the difference being that both the length and the cross-section are those of some intermediate line field and cross-section. For the case of the sphere one may use it to obtain the following upper and lower bounds:

$$\begin{aligned}
&2a \le L(c_2,c_3) \le \pi a, \quad \pi b^2 \le A(c_1) \le \pi a^2, \\
&\frac{2a}{\pi \sigma a^2} = \frac{2}{\pi \sigma a} \le R \le \frac{\pi a}{\pi \sigma b^2} = \frac{a}{\sigma b^2}.
\end{aligned} \tag{64}$$

When the potential is known, the intervening magnitudes may be evaluated using eqs. (9), (15), (61) and (62):

$$\begin{aligned}
L(c_2,c_3) &= \frac{\Omega}{I} = \frac{\iiint |\nabla \phi| \, dv}{\left| \iint_S \nabla \phi \bullet d\vec{S} \right|}, \\
A(c_1) &= \frac{\Omega}{\sigma \cdot V} = \frac{\iiint |\nabla \phi| \, dv}{\phi_f - \phi_0}.
\end{aligned} \tag{65}$$

The cumbersome calculation of the numerator is not necessary for evaluating the resistance, which may be obtained only from the denominators. Eq. (65) is only given here to show that $L(c_1)$ and $A(c_2,c_3)$ may be evaluated if one wishes to better understand the factors that lead to a particular value of electrical resistance.

Conclusions

The equations and boundary conditions that govern the flow of current under applied voltages have been thoroughly discussed, showing that the problem requires solving Laplace's equation with apropriate boundary conditions. An explicit calculation was made of the equipotential surfaces and field-current lines for the case of a sphere, and its resistance was calculated under quite reasonable assumptions. The calculation shows that the boundary conditions on the external surface and electrodes of a selected piece of conductor cannot always be met and that the value of the electrical resistance R depends critically on the rest of the circuit. The understanding of the main factors that determine R paved the way for the next step, the general discussion of the linear relationship between current I and potential difference V.

With the help of a volume integral of J expressed in suitable curvilinear coordinates, and a mean value theorem of integral calculus, the linear relationship between V and I —resistance R— may be generally expressed in terms of some effective length and cross-section of the chosen piece of circuit. Although the expression is not necessarily appropiate for determining the actual values of R —for which gives only gross lower and upper bounds— it explicitly puts into evidence the main factors that govern its value.

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Up to this version no peer review contribution has been made to the present work.