Letters

Magnetostatics of Anisotropic Ellipsoidal Bodies

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Abstract—**The demagnetizing tensor formalism for the evaluation of magnetization and internal field in ellipsoidal bodies of isotropic materials is extended to the case of anisotropic materials and to the calculation of the field outside the body.**

Ellipsoids are the only known nonhysteretic finite bodies that become homogeneously magnetized when placed in a uniform applied field. The case of homogeneous and isotropic materials thus shaped has been discussed either in the linear range or in the saturation limit [1]–[3]. In what follows we will analyze the case of homogeneous but anisotropic ellipsoids for the general nonlinear situation. Instead of using the cumbersome method of separation of variables in ellipsoidal coordinates, an integral equation will be written for H. A certain vectorial function will then be shown to satisfy the equation, thus being the sought after solution.

Let H_0 be the uniform applied magnetic field with fixed sources, H the macroscopic magnetic field, and M the body's magnetization. Then [4]

$$H(r) = H_0 + \frac{\lambda}{4\pi} \nabla \int_V M(r') \cdot \nabla \left(\frac{1}{|r - r'|} \right) d^3 r', \tag{1}$$

where ∇ operates on r, and the integration is over the body's volume V. (For S.I. units set $\lambda = 1$, and for Gaussian units set $\lambda = 4\pi$.)

In the general nonhysteretic anisotropic case, M will be a single-valued vector function of H,

$$M(r) = X(H(r)). \tag{2}$$

It then follows that

$$H(r) = H_0 + \frac{\lambda}{4\pi} \nabla \left(\nabla \cdot \int_V \frac{X(H(r'))}{|r - r'|} d^3 r' \right), \tag{3}$$

where a simple property of ∇ has been used. Equation (3) is the integral equation that uniquely determines H. We now try the following solution,

$$H(r) = H_0 - \lambda d(r) \cdot C, \tag{4}$$

where C is a constant vector, and $d(r)$ is a generalized depolarization tensor [5] whose components in Cartesian coordinates x_1, x_2, x_3 are given by

$$d_{jk}(r) = -\frac{1}{4\pi} \frac{\partial^2}{\partial x_j \partial x_k} \int_V \frac{d^3 r'}{|r - r'|}. \tag{5}$$

When r is an interior point, d becomes the usual demagnetizing tensor D whose properties have been fully discussed [3], [6], and [7]. We use the convention

$$\text{trace } D = 1. \tag{6}$$

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As D is constant for ellipsoids, when using (4) X becomes a constant vector for interior points and may thus be taken outside the integral sign in (3). Therefore

$$H(r) = H_0 + \frac{\lambda}{4\pi} \nabla \left(\nabla \cdot \left(X(H_0 - \lambda D \cdot C) \int_V \frac{d^3 r'}{|r - r'|} \right) \right)$$

$$= H_0 - \lambda d(r) \cdot X(H_0 - \lambda D \cdot C), \tag{7}$$

where in the last step use has been made of (5). From (4), (7), and (2) it is seen that

$$C = M = X(H_0 - \lambda D \cdot M). \tag{8}$$

Therefore, due to the fact that the demagnetizing tensor is constant for ellipsoids, it follows that the magnetization and the interior field are also constant. Notice that, when replacing H_0 by an adequate effective field, (8) is the starting point in the molecular field approximation for a single-domain ferromagnet, where X is usually taken to be proportional to the Brillouin function [8]. In the linear range (2) may be approximated by

$$M = \boldsymbol{\chi} \cdot H, \tag{9}$$

where $\boldsymbol{\chi}$ is the magnetic susceptibility tensor. Then (8) may be solved for M giving

$$M = \frac{1}{V} \boldsymbol{\alpha} \cdot H_0, \tag{10}$$

$$m = \boldsymbol{\alpha} \cdot H_0, \tag{11}$$

where m is the body's total magnetic dipole moment and $\boldsymbol{\alpha}$ is the body's magnetic polarizability tensor

$$\boldsymbol{\alpha} = V(1 + \lambda \boldsymbol{\chi} \cdot D)^{-1} \cdot \boldsymbol{\chi} = V(\boldsymbol{\chi}^{-1} + \lambda D)^{-1}. \tag{12}$$

The polarizability tensor that has been previously defined only for superconducting ellipsoids [9] provides a compact expression for the torque τ impressed on the body by the applied field,

$$\tau = \mu_0 m \times H_0 = \mu_0 (\boldsymbol{\alpha} \cdot H_0) \times H_0. \tag{13}$$

As soon as M is determined, either from (8) or from the linear approximation (10), the magnetic field is also determined both inside and outside the ellipsoidal body as long as $d(r)$ is known. The constant principal values of D (interior d) have already been tabulated [3], [6]. The exterior values of the depolarization tensor d may be found in a similar way, deriving them from the integral

$$I(r) = \int_V \frac{d^3 r'}{|r - r'|} \tag{14}$$

through (5). $I(r)$ is the electrostatic potential of a uniformly charged ellipsoid, and has been calculated previously in the general case [10], [11].

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Campos eléctricos generados por elipsoides uniformemente polarizados

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Se expresan los campos eléctricos $\vec{E}(\vec{r})$, dentro y fuera de elipsoidales uniformemente polarizados, en términos de integrales elípticas y sin necesidad de resolver ecuaciones diferenciales. Las expresiones son válidas para materiales homogéneos cualesquiera, sean isótropos o anisótropos, sean electretos, dieléctricos o conductores. Se dan las expresiones explícitas del campo inducido $\vec{E}(\vec{r})$ para esferas dieléctricas y conductoras inmersas en campos aplicados constantes y uniformes.

Descriptores: Electrostática; ecuaciones de Poisson y Laplace; problemas de límite-valor.

The electric fields $\vec{E}(\vec{r})$, inside and outside uniformly polarised ellipsoidal electrets, dielectrics and conductors, is given in terms of elliptic integrals. The derivation, valid for homogeneous isotropic and anisotropic materials, makes no recourse to differential equations. The full expression of the $\vec{E}(\vec{r})$ induced for spherical bodies embedded in uniform aplied constant electric fields, either dielectrics or conductors, is explicity given.

Keywords: Electrostatics; Poisson and Laplace equations; boundary-value problems.

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1. Introducción

Los libros introductorios de electricidad rara vez discuten el problema de la electrificación de cuerpos finitos. La razón es la dificultad del cálculo de distribuciones de campos eléctricos generados por materiales cuyo estado de polarización depende de la misma configuración final de estos campos. Por tratarse de un problema que debe resolverse de modo autoconsistente (la polarización de una porción de materia depende de los campos de la materia restante y ésta depende a su vez de los campos que genera esa porción), se deja usualmente para cursos avanzados donde se resuelven las ecuaciones diferenciales a derivadas parciales de Maxwell por el método de separación de variables. Entre los pocos cuerpos detalladamente discutidos en los textos elementales se cuentan las láminas de espesor constante y extensión infinita y los cilindros rectos de sección circular y longitud infinita, siempre para materiales homogéneos e isótropos. La esfera, el cuerpo finito más simple, sólo se resuelve con ecuaciones diferenciales. Resulta entonces que sólo en los cursos más avanzados de electromagnetismo, usualmente no tomados por ingenieros, es posible trabajar con cuerpos reales (finitos) y discutir importantes comportamientos de interés técnico, como el análisis de las condiciones en que su polarización puede ser uniforme, los efectos de la anisotropía (materiales cristalinos) y el comportamiento de conductores finitos en presencia de campos aplicados uniformes.

La principal razón de la dificultad para resolver casos más realistas es que los cursos introductorios de electricidad se dictan usualmente antes de que los estudiantes conozcan los métodos del análisis vectorial. La situación está cambiando en las universidades donde los cursos de electricidad y magnetismo se dictan inmediatamente a continuación de los de análisis vectorial. Esto permite tanto dar demostraciones generales rigurosas de las propiedades electromagnéticas de la materia (en vez de los usuales casos especiales que sólo ilustran la plausibilidad de las leyes invocadas) como resolver sin necesidad de ecuaciones diferenciales el único caso conocido de cuerpos finitos cuya polarización eléctrica es uniforme, los elipsoides generales, problema este último que resolvemos detalladamente en este trabajo.

2. Cuerpos con polarización permanente

El potencial eléctrico V generado por un volumen v de dieléctrico con polarización uniforme $\vec{P}$ es

$$V(\vec{r}) = k_1 \iiint_v \frac{\vec{P}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d^3\vec{r}$$

$$= k_1\, \vec{P}\cdot \iiint_v \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d^3\vec{r}, \qquad (1)$$

donde $k_1 = 4\pi/\varepsilon_0 = 10^{-7}c^2$ en el Sistema Internacional (SI)[1], donde c es la velocidad de propagación de las ondas electromagnéticas en el vacío y $d^3\vec{r}$ es el elemento diferencial de volumen. Como

$$\nabla \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3},$$

donde el gradiente se toma respecto de la variable vectorial $\vec{r}$, se puede reescribir (1) de la forma:

$$V(\vec{r}) = -\hat{P}\cdot\nabla\phi(\vec{r}), \quad \text{donde}$$

$$\phi(\vec{r}) = k_1 \iiint_v \frac{P}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}, \qquad (2)$$

donde $\hat{P}$ es el versor adimensional en la dirección y sentido de $\vec{P}$, $\hat{P}\cdot\nabla$ es el operador derivada en la dirección $\hat{P}$ y $\phi(\vec{r})$

es —salvo una diferencia de unidades[i]—el potencial eléctrico generado cuando el volumen v está cargado con densidad de carga uniforme $P/[l]$[ii]. Este mismo resultado puede obtenerse mediante la superposición de dos distribuciones de carga uniforme Q idénticas cuya distancia a se hace tender a 0 mientras se mantiene constante el producto $Q \cdot a = P$ (definición matemática de dipolo puntual).

De la relación entre campo y potencial eléctrico se tiene que

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = -\nabla \left[(\hat{P} \cdot \nabla)\phi(\vec{r}) \right]. \qquad (3)$$

Para simplificar la escritura de las ecuaciones posteriores definimos

$$I(\vec{r}) = \frac{\phi(\vec{r}) \cdot [l]}{k_1 P} = \iiint_v \frac{1}{|\vec{r} - \vec{r}'|}\, d^3\vec{r}', \qquad (4)$$

lo que permite simplificar mucho los cálculos al remitirnos al problema más simple del cálculo de ϕ. Explicitando las derivadas se obtiene finalmente

$$\vec{E}(\vec{r}) = k_1 \sum_\alpha \hat{x}_\alpha \frac{\partial}{\partial x_\alpha} \sum_\beta \frac{\partial I(\vec{r})}{\partial x_\beta}\, \hat{x}_\beta \cdot \vec{P}$$

$$= k_1 \sum_\alpha \hat{x}_\alpha \sum_\beta \frac{\partial^2 I(\vec{r})}{\partial x_\alpha \partial x_\beta}\, P_\beta, \qquad (5)$$

donde $\hat{x}_\beta$ es el versor del eje coordenado x_β, con $x_\beta = x$, y, z. Las componentes del campo eléctrico son entonces[2]

$$E_\alpha(\vec{r}) = -4\pi k_1 \sum_\beta n_{\alpha\beta}\, P_\beta, \quad \text{donde}$$

$$n_{\alpha\beta}(\vec{r}) = -\frac{1}{4\pi} \frac{\partial^2 I(\vec{r})}{\partial x_\alpha \partial x_\beta}. \qquad (6)$$

El tensor adimensional de componentes $\mathrm{n}_{\alpha\beta}$ fue originalmente introducido para la resolución de problemas de magnetización[3] donde se lo denomina *tensor demagnetización*. Su aplicabililidad a problemas tanto de polarización eléctrica como magnética, donde caracteriza los efectos de depolarización provenientes de la forma del cuerpo, justifica el uso de un nombre más general, como el de **tensor depolarización** que se usará aquí. Se introduce el factor –1/4π en la definición para que su traza valga la unidad (véase la sección **Propiedades del tensor depolarización**).

Es más fácil calcular $\vec{E}(\vec{r})$ en notación matricial. Para ello se reescriben los vectores como matrices columna y el tensor depolarización como una matriz cuadrada, dando

$$\vec{E}(\vec{r}) = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

$$\mathbf{n}(\vec{r}) = \begin{pmatrix} n_{xx}(\vec{r})\ n_{xy}(\vec{r})\ n_{xz}(\vec{r}) \\ n_{yx}(\vec{r})\ n_{yy}(\vec{r})\ n_{yz}(\vec{r}) \\ n_{zx}(\vec{r})\ n_{zy}(\vec{r})\ n_{zz}(\vec{r}) \end{pmatrix},$$

$$\vec{P} = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \qquad (7)$$

En esta notación la Ec. (6) se escribe simplemente de la forma $\vec{E}(\vec{r}) = -4\pi\, k_1 \mathbf{n}(\vec{r}) \cdot \vec{P}$, o más explícitamente,

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -4\pi k_1 \begin{pmatrix} n_{xx}(\vec{r})\ n_{xy}(\vec{r})\ n_{xz}(\vec{r}) \\ n_{yx}(\vec{r})\ n_{yy}(\vec{r})\ n_{yz}(\vec{r}) \\ n_{zx}(\vec{r})\ n_{zy}(\vec{r})\ n_{zz}(\vec{r}) \end{pmatrix} \cdot \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \qquad (8)$$

Se ve así que el problema del cálculo del campo eléctrico $\vec{E}(\vec{r})$ producido por un volumen v de materia con polarización uniforme $\vec{P}$ se ha reducido al cálculo del tensor depolarización $\mathbf{n}(\vec{r})$. Este tensor se obtiene a partir del potencial generado por una densidad de carga $\rho = P/[l]$ uniformemente distribuida en el volumen v. En todos los casos donde este potencial pueda expresarse analíticamente (caso de los elipsoides generales), se puede también expresar el campo eléctrico generado por ese volumen cuando tiene polarización eléctrica uniforme.

3. Propiedades del tensor de polarización

Se dan a continuación, sin demostración, propiedades de **n** que pueden deducirse de su definición[4]:

- El tensor depolarización es simétrico: $\mathrm{n}_{\alpha\beta} = \mathrm{n}_{\beta\alpha}$.
- La traza del tensor depolarización vale 1 dentro del volumen v y 0 afuera:

$$\mathrm{Tr\ n} = \mathrm{n}_{xx}(\vec{r}) + \mathrm{n}_{yy}(\vec{r}) + \mathrm{n}_{zz}(\vec{r}) = \begin{cases} 1 \text{ si } \vec{r} \in v \\ 0 \text{ si } \vec{r} \notin v \end{cases}.$$

- Cuando el volumen v es un elipsoide general[5]:
  - Los valores de las componentes del tensor depolarización, que denominamos N, son constantes para todos los puntos interiores a v. No sucede lo mismo en los puntos exteriores a v, donde $\mathbf{n}(\vec{r})$ no es constante [Ec. (12)]. Los valores de N derivados de la Ec. (6) pueden expresarse en términos de integrales elípticas[6].
  - De la Ec. (8) se ve que el campo eléctrico generado por la polarización es uniforme en el interior de elipsoides, pero que ambos vectores no son en general paralelos. Este fenómeno, bien conocido en el campo de experimentos con materia magnéticamente polarizada, se denomina anisotropía de forma.
  - El tensor depolarización interior **N** es diagonal en el sistema cartesiano de coordenadas coincidente con los ejes principales del elipsoide de semiejes a_1, a_2 , a_3 y ecuación

$$(x/a_1)^2 + (y/a_2)^2 + (z/a_3)^2 = 1.$$

  - Los términos diagonales son entonces, como se demuestra en cualquier curso de teoría de matrices, los autovalores de **N**.

- Si dos semiejes del elipsoide son iguales, los autovalores correspondientes de **N** también lo son.
- Cuando un semieje tiende a ∞, el correspondiente autovalor de **N** tiende a 0.

4. Tensor depolarización de una esfera

Se calcula a continuación el tensor depolarización de la esfera, el cuerpo finito de máxima simetría. Para ello se evalúa el potencial ϕ dado por la Ec. (2). Si el radio de la esfera es R, su carga total es Q y se toma el origen del sistema de coordenadas cartesianas en el centro de la esfera, del teorema de Gauss de la electrostática se obtiene[7]:

$$E(r) = \begin{cases} \dfrac{k_1 Q}{R^3} r & \text{si} \quad r \leq R, \\ \dfrac{k_1 Q}{r^2} & \text{si} \quad r \geq R. \end{cases} \tag{9}$$

El campo eléctrico es radial y de sentido saliente de la esfera cuando Q es positivo. Su módulo $E(r)$ es función sólo del módulo r del vector posición $\vec{r}$. Como

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = -\frac{\partial V(r)}{\partial r}\,\hat{r} = E(r)\,\hat{r},$$

se puede integrar $\partial V(r)/\partial r = -E(r)$ para obtener

$$V(r) = \begin{cases} -\dfrac{k_1 Q}{2R^3} r^2 & \text{si} \quad r \leq R, \\ \dfrac{k_1 Q}{r} & \text{si} \quad r \geq R. \end{cases} \tag{10}$$

Teniendo en cuenta que $v = (4\pi/3)R^3$, $Q = \rho \cdot v$, $\rho = P/[l]$, $Q = (4\pi/3)R^3(P/[l])$, donde v es el volumen de la esfera, se obtiene, de la Ec. (4),

$$I(r) = \begin{cases} -\dfrac{2\pi}{3} r^2 & \text{si} \quad r \leq R, \\ \dfrac{v}{r} & \text{si} \quad r \geq R. \end{cases} \tag{11}$$

Utilizando la definición de **n** [Ec. (6)] se obtiene finalmente

$$\begin{aligned} \mathbf{N}(\vec{r}) &= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \frac{1}{3}\mathbf{1}, \\ \mathbf{n}(\vec{r}) &= -\frac{v}{4\pi} \begin{pmatrix} \frac{3x^2-r^2}{r^5} & \frac{3x\cdot y}{r^5} & \frac{3y\cdot z}{r^5} \\ \frac{3x\cdot y}{r^5} & \frac{3y^2-r^2}{r^5} & \frac{3z\cdot x}{r^5} \\ \frac{3y\cdot z}{r^5} & \frac{3z\cdot x}{r^5} & \frac{3z^2-r^2}{r^5} \end{pmatrix}, \end{aligned} \tag{12}$$

donde **N** es el tensor depolarización interior ($r \leq R$), **1** es la matriz unidad y **n** el tensor depolarización exterior ($r \geq R$). Nótese que Tr **N** = 1 y Tr **n** = 0, como corresponde a las propiedades generales enunciadas. El campo eléctrico exterior a la esfera es exactamente el de un dipolo puntual con momento dipolar eléctrico $\vec{p} = v \cdot \vec{P}$.

Usando las Ecs. (8) y (12) se obtiene la expresión del campo eléctrico $\vec{E}$ generado por una esfera con polarización uniforme $\vec{P}$ cuando no hay aplicado un campo externo. Para que la expresión resultante resulte familiar se ha remplazado la polarización eléctrica $\vec{P}$ por el momento dipolar eléctrico $\vec{p} = v \cdot \vec{P} = (4\pi/3)R^3\vec{P}$, obteniéndose finalmente

$$\vec{E}(\vec{r}) = -\frac{4\pi\, k_1}{v}\mathbf{n}(\vec{r})\cdot\vec{p} = \begin{cases} -\dfrac{4\pi k_1}{3}\vec{p} & \text{si} \quad r \leq R, \\ -4\pi k_1 \begin{pmatrix} \frac{3x^2-r^2}{r^5} & \frac{3x\cdot y}{r^5} & \frac{3y\cdot z}{r^5} \\ \frac{3x\cdot y}{r^5} & \frac{3y^2-r^2}{r^5} & \frac{3z\cdot x}{r^5} \\ \frac{3y\cdot z}{r^5} & \frac{3z\cdot x}{r^5} & \frac{3z^2-r^2}{r^5} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} & \text{si} \quad r \geq R \end{cases} . \tag{13}$$

Para $r \leq R$ el campo obtenido corresponde al llamado campo de Lorentz[8], cuyo valor se calcula usualmente mediante la expresión integral del campo eléctrico para densidades superficiales de cargas de polarización. Usando la relación

$$(3x^2 - r^2)p_x + 3x\,y\,p_y + 3x\,z\,p_z = 3x\,\vec{p}\cdot\vec{r} - r^2 p_x$$

y las ecuaciones análogas para cada fila del producto matricial $\mathbf{n}\cdot\vec{p}$, se puede reescribir la expresión del campo exterior exclusivamente en término de vectores:

$$\vec{E}(\vec{r}) = -\frac{4\pi k_1}{v}\mathbf{n}(\vec{r})\cdot\vec{p} = k_1\frac{3\vec{r}\,(\vec{p}\cdot\vec{r}) - r^2\vec{p}}{r^5} \quad si \quad r \geq R. \tag{14}$$

Este campo corresponde exactamente al de un dipolo puntual de momento dipolar eléctrico $\vec{p}$ situado en el origen de coordenadas[9].

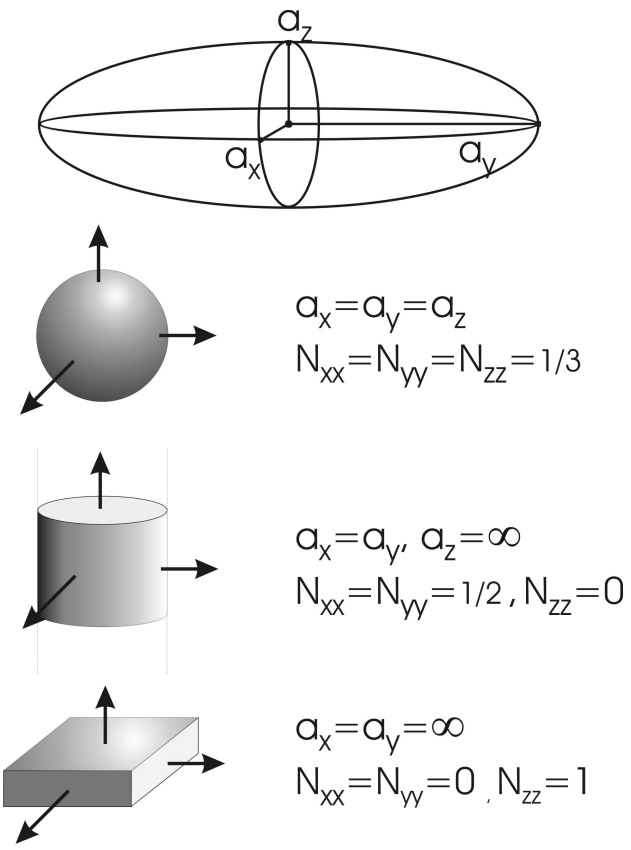

FIGURA 1.

5. Cálculo de n a partir de sus propiedades

En casos de alta simetría, o cuando alguno de los semiejes tiende a ∞, el valor del tensor depolarización interior **N** puede obtenerse sin necesidad de evaluar ninguna integral. Para ello basta utilizar algunas de las propiedades antes enunciadas, todas cumplidas por la Ec. (12):

- **N** es diagonal cuando el sistema de coordenadas cartesiano coincide con los ejes principales del elipsoide, cuyos semiejes denotamos con a_x, a_y y a_z.
- La traza de **N** vale siempre 1.
- Si dos semiejes del elipsoide son iguales, los autovalores correspondientes de **N** también lo son.
- Cuando el valor de un semieje tiende a ∞, el correspondiente autovalor de **N** tiende a 0.

Usando estas propiedades se pueden calcular fácilmente los valores de **N** correspondientes a una esfera, un cilindro infinitamente largo de sección circular y una lámina de espesor finito y extensión indefinida. Los resultados se ilustran en la Fig. 1.

Es importante señalar que cuando uno o más semiejes divergen el tensor depolarización interior **N** sólo puede evaluarse tomando límites en la integral elíptica general (no dada aquí) o usando las propiedades de simetría precedentes. No es correcto en este caso usar el potencial derivado del teorema de Gauss de la electrostática. Como ejemplo de la aplicación de las propiedades señaladas, se calcula a continuación el tensor depolarización interior de una lámina dieléctrica de extensión indefinida y espesor finito d, donde se toma como z el eje normal a la interfase dieléctrico-vacío. Como los semiejes a_x y a_y tienden a ∞, se tiene que $N_{xx} = N_{yy} = 0$. Por la regla de la traza se obtiene finalmente que

$$N_{xx} + N_{yy} + N_{zz} = 1, \quad \text{es decir,} \quad N_{zz} = 1.$$

No hay ninguna propiedad general que permita simplificar de manera análoga el cálculo del tensor depolarización exterior. En este caso las únicas propiedades disponibles son

- $n_{\alpha\beta}(\mathbf{r}) = n_{\beta\alpha}(\mathbf{r})$;
- Tr **n** = 0.

Usando estas propiedades el número de componentes a evaluarse se reduce a 5, derivándose de éstas las cuatro restantes.

6. Polarización inducida

Por ser el conceptualmente más simple, se ha discutido hasta ahora sólo el caso de polarizaciones eléctricas permanentes, es decir, el de ferroeléctricos o electretos. En el caso de dieléctricos normales la polarización es generada por el mismo campo eléctrico aplicado. En el rango lineal se tiene entonces

$$\vec{P}(\vec{r}) = \chi \cdot \vec{E}(\vec{r}), \tag{15}$$

donde, salvo en el caso isótropo, la matriz susceptibilidad eléctrica χ no es múltiplo de la matriz unidad **1** (matriz no diagonal o diagonal con autovalores diferentes). Se considera aquí sólo el caso homogéneo (χ es una matriz constante independiente de las coordenadas) cuando se aplica un campo $\vec{E}_0$ uniforme en todo el volumen v. De la primera de las Ecs. (8) se obtiene entonces que para los puntos interiores al elipsoide

$$\vec{E}(\vec{r}) = \vec{E}_0 - 4\pi\, k_1 \mathbf{N} \cdot \chi \cdot \vec{E}(\vec{r}),$$

ecuación matricial (o sistema de ecuaciones lineales) que puede resolverse para el campo $\vec{E}(\vec{r})$. Se ve que el campo eléctrico interior es uniforme en v y satisface la ecuación

$$(\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi) \cdot \vec{E} = \vec{E}_0, \quad \text{que invertida da}$$

$$\vec{E} = (\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi)^{-1} \cdot \vec{E}_0, \tag{16}$$

donde **1** es la matriz unidad y $(\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi)^{-1}$ es la matriz inversa de $(\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi)$. A partir de la Ec. (16) se obtiene la expresión completa del campo eléctrico de un elipsoide dieléctrico de susceptibilidad χ colocado en un campo aplicado uniforme $\vec{E}_0$. El campo eléctrico interior genera una polarización, también uniforme,

$$\vec{P} = \chi \cdot \vec{E} = \chi \cdot (\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi)^{-1} \cdot \vec{E}_0. \tag{17}$$

Esta polarización genera a su vez el campo exterior al elipsoide dado por Ec. (8), al que debe sumarse el aplicado, dando finalmente

$$\vec{E}(\vec{r}) = \vec{E}_0 - 4\pi\, k_1 \mathbf{n}(\vec{r}) \cdot \vec{P}$$
$$= [\mathbf{1} - 4\pi\, k_1 \mathbf{n}(\vec{r}) \cdot \chi \cdot (\mathbf{1} + 4\pi k_1 \mathbf{N} \cdot \chi)^{-1}] \cdot \vec{E}_0, \quad (18)$$

donde la dependencia espacial (pero no la orientación cuando χ no es múltiplo de la matriz unidad) está exclusivamente determinada por $\mathbf{n}(\vec{r})$.

Es importante discutir ahora la hipótesis de polarización uniforme hecha al comienzo del trabajo. Cuando a un dieléctrico de forma arbitraria se le aplica un campo uniforme $\vec{E}_0$, la polarización inducida no será en general uniforme. Esto se debe a que el campo inductor en cualquier molécula del material es la composición del campo aplicado y el campo dipolar $\vec{E}_{\vec{P}}$ generado por las restantes moléculas. Maxwell demostró que $\vec{E}_{\vec{P}}$ sólo puede ser uniforme cuando la superficie límite del dieléctrico está descrita por una expresión algebraica de segundo grado en las coordenadas x, y, z[10]. La única superficie cerrada de este tipo es el elipsoide general, mientras que los casos límites de semiejes infinitos corresponden a superficies abiertas[iii]. Ésta es la razón por la cual el caso más simple en que un dieléctrico puede tener polarización uniforme es cuando tiene forma elipsoidal, y su polarización está entonces dada por la Ec. (17).

7. Esfera conductora en campo aplicado uniforme

La redistribución de cargas en conductores puede obtenerse como el caso límite en que la polarización de la materia puede producirse con total libertad de desplazamiento de las cargas eléctricas. En tal caso, todas las cargas se ubican en la superficie y ninguna en el interior. La polarización que resulta equivalente $\vec{P}$ puede calcularse sabiendo que en el estado de equilibrio en el interior del conductor el campo resultante debe ser nulo. Es decir, el campo generado por la distribución superficial de cargas debe cancelar exactamente el campo aplicado. Si $\vec{E}_0$ es el campo uniforme aplicado, se tiene entonces, por la adición de este campo al de polarización descrito por la Ec. (8)

$$\vec{E}(\vec{r}) = \vec{E}_0 - 4\pi\, k_1 \mathbf{N} \cdot \vec{P} = 0, \quad \text{que permite despejar}$$
$$4\pi k_1 \mathbf{N} \cdot \vec{P} = \vec{E}_0. \quad (19)$$

Se obtiene $\vec{P}$ resolviendo este sistema de ecuaciones lineales, que es equivalente a invertir la matriz **N**. Para una esfera metálica de radio R, donde **N** = **1**/3, la Ec. (19) da

$$\vec{P} = \frac{3}{4\pi k_1}\vec{E}_0, \quad \vec{p} = v \cdot \vec{P} = \frac{R^3}{k_1}\vec{E}_0, \quad (20)$$

donde $\vec{p}$ es el momento dipolar eléctrico de la esfera. El campo resultante en el exterior es la composición del campo aplicado con el campo dipolar dado por la Ec. (14). Se puede asimismo calcular la densidad superficial de carga σ inducida sobre la superficie de la esfera usando la bien conocida fórmula

$$\sigma(\vec{R}) = \hat{R} \cdot \vec{P}, \quad (21)$$

donde $\hat{R}$ es el versor adimensional normal y saliente de la superficie de la esfera en el punto $\vec{R}$. Si se elige el sistema de coordenadas de modo que $\vec{P}$ coincida con el eje z y θ es el ángulo que forma $\hat{R}$ con z, se obtiene $\sigma(\theta) = P{\cdot}\cos(\theta)$. Esto muestra que, como debe ser, σ alcanza su valor máximo positivo en (0, 0, R), su máximo negativo en (0, 0, R) y se anula sobre el plano ecuatorial z = 0.

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i. La diferencia consiste en que el numerador debiera tener unidades de densidad de carga $[q]{\cdot}[l]^3$ (C/m^3 en SI) y tiene en cambio unidades de densidad de polarización $[q]{\cdot}[l]^2$ (C/m^2 en SI).

ii. $[l]$ es, en notación internacional, la designación de la unidad de longitud del sistema en uso.

iii. Según el conocimiento del autor, no parece haber sido estudiado el caso en que el cuerpo está delimitado por la intersección de dos o más superficies diferentes de segundo grado.

1. Los correspondientes a otros sistemas de unidades pueden obtenerse de John David Jackson, *Classical electrodynamics* (John Wiley & Sons, New York 1962) p. 616.
2. La versión magnética de este resultado fue publicado por primera vez por el autor en *IEEE Trans. Magn.* **17** (1981) 1363.
3. D. Landau y E.M. Lifshitz, *Electrodynamics of continuous media* (London 1941), p. 26.
4. R. Moskowitz y E. Della Torre, *IEEE Trans. Magn.* **2** (1966) 739 y referencias allí dadas.
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Derivation of Analytical Expressions for the Magnetic Torque as a Function of Experimental Parameters

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Abstract—**Simple analytical expressions are derived for the magnetic torque on a magnetic crystal as a function of field intensity and orientation. The expressions, obtained by means of a recently introduced method of analytical continuation of series, are accurate enough in the whole range of physical parameters. This makes it possible to use them as universal curves for describing experimental results.**

I. Introduction

EXPERIMENTS based on measurements of magnetic torque are an important source to determine some physically meaningful information on magnetic crystalline solids. The saturation magnetization and the magnetic anisotropy constants are, for instance, some of the relevant physical parameters [1], [2].

In order to obtain such constants from torque experiments, it is necessary to have analytical expressions for the torque (or other properties related to it) as a function of the experimental variables. In our case there are two such variables: the intensity of a static magnetic field applied on the crystal (H) and the angle between that field and a properly chosen crystal axis (θ).

Several procedures have been developed to get such analytical expressions [3], [4], all of them based on representing the torque as a Fourier series in $\sin\theta$ with coefficients depending on H. These series involve a sort of strong-field approximation which limits their range of applicability. In fact, it is at present not quite clear if the expressions derived in that way can be applied in a wide range of values of H and θ.

If one looks for a representation of the magnetic torque as an expansion in power series of H, a new difficulty arises owing to the lack of convergence of the series in the strong field case. The Fourier series does not seem to be a better alternative, because, as each Fourier coefficient generates all powers of H, the truncation of the former cannot be compared to any truncated expansion in powers of H [1], [3], [4].

The aim of this paper is to apply a new method of handling a power series in order to get an analytic universal curve of magnetic torque valid for both intense and low magnetic fields. To avoid the inconveniences of the aforementioned methods, we must start from new principles. With that purpose we will take advantage of a method, called from now on the functional method (FM) [5]–[7], recently developed to deal with a wide class of physical problems posed by power series. This technique has been successful in obtaining simple approximate analytical expressions for the function of interest, by using both a number of its Taylor coefficients and information about the asymptotic behavior of the function.

The paper has been organized as follows: in Section II we discuss the magnetic-anisotropy energy for various crystals and obtain some analytical information about the magnetic torque as a function of experimental parameters. In Section III we apply the FM to derive a universal analytical curve for the magnetic torque. The present version of the FM differs from the previous one [5]–[7], because of the presence of a second variable (besides H) in the angle θ. Results and some experimental consequences of our procedure are discussed at length in Section IV.

II. Magnetic-Anisotropy Energy and Magnetic Torque

Let us briefly review the main results referring to the magnetic torque of a crystal, in order to obtain the equations which are the object of this study. To discuss the basic relationships, it is enough to consider at present only a cubic crystal. Let $\boldsymbol{M}$ be the intrinsic magnetization of the crystal; for such a system the magnetic anisotropy energy (per unit volume) is given by [1], [2]

$$E = K_1(a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2) + K_2a_1^2a_2^2a_3^2 \quad (1)$$

where $\{a_i\}$ are the direction cosines of $\boldsymbol{M}$, and K_1, K_2 are known as the magnetic-anisotropy constants. In order to measure the magnetic torque, the crystal is allowed to rotate around a given axis. In order to apply the method later to a larger class of crystals, we will consider here a convenient choice of rotation axis; if we now choose the crystal direction $|001|$ as that axis, (1) becomes [1], [2]

$$E = K_1 \cos^2\alpha \sin^2\alpha \quad (2)$$

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where $a_1 = \cos\alpha$, $a_2 = \sin\alpha$, and $a_3 = 0$. Using (2), we get for the magnetic torque

$$\tau = \partial E/\partial\alpha = (K_1/2)\sin 4\alpha. \tag{3}$$

If we apply an external magnetic field $\boldsymbol{H}$ on the crystal, with an angle θ with respect to the axis $|100|$, we obtain a total torque

$$\begin{aligned}\tau_t &= \tau - \partial/\partial\alpha(\boldsymbol{H}\cdot\boldsymbol{M}) \\ &= \tau - \partial/\partial\alpha\left\{HM\cos(\theta-\alpha)\right\}, \\ H &= \|\boldsymbol{H}\|, \quad M = \|\boldsymbol{M}\|\end{aligned} \tag{4}$$

where the equilibrium condition (i.e., $\tau_t = 0$) allows one to determine τ experimentally. Our problem is, of course, to obtain the physically meaningful magnetic parameters of the crystal from the measurement.

In the equilibrium condition, (4) can be rewritten as

$$\tau' = \sin 4\alpha = b\sin(\theta-\alpha) \tag{5}$$

where $\tau' = \tau/K_1$ and $b = 2HM/K_1$ are dimensionless constants. In particular, $M_\perp = K_1\tau'/2H$ is the normal component of the magnetization. The parameters θ and H are known beforehand, while α, K_1, and M are unknown and must be determined.

It should be stressed that (5) is general enough for our purposes. Although only K_1 appears in (5), an appropriate experimental dispositive allows one to "uncouple" K_2 from K_1, and to describe the former through an equation similar to (5) [1, p. 187]. Furthermore, (5) stands also for the third magnetic anisotropy constant in a tetragonal axis (see details in Section IV).

In order to obtain the physical parameter characterizing the system, it is necessary to have accurate analytical expressions for the magnetic torque as a function of b and θ. The main purpose of this paper is the derivation of such expressions using a new method. As we shall see, it is designed to avoid the main drawbacks characteristic of the procedures based on the Fourier series [1], [3], [4].

From simple trigonometric relationships, (5) can be transformed into an algebraic equation

$$\begin{aligned}&-64x^8 + 128x^6 - 16bx^5\sin\theta - 80x^4 \\ &\quad + 24x^3\sin\theta + (16-b^2)x^2 - 8bx\sin\theta \\ &\quad + b^2\sin^2\theta = 0, \qquad x = \sin\alpha.\end{aligned} \tag{6}$$

Let us first analyze the number and type of roots of this equation in order to obtain analytical expressions for x in terms of $\sin\theta$ and b.

When $b = 0$ we have the roots: $x^{(0)} = x(b=0) = 0$, $1/2$, $-1/2$, 1, and -1, whose multiplicities are 2, 2, 2, 1, and 1, respectively. Among them, $x^{(0)} = \pm 1$ correspond to the case where $\boldsymbol{M}$ coincides with a crystal axis ($\alpha = \pm\pi/2$). The remaining roots are at the same time local extrema. On the other hand, when $b \to \infty$ we have $x'^{(0)} = x(1/b = 0) = \pm\sin\theta$, that is, only two real roots. Owing to the periodic nature of the problem, the range $\pi/4 \le \theta \le \pi/2$ is enough for all purposes, which shows that the important roots are those which correlate as

$$\lim_{b\to 0} x = 1 \qquad \lim_{b\to\infty} x = \sin\theta. \tag{7}$$

Equation (7) can be easily understood: when the magnetic field intensity is zero ($b = 0$), the magnetization $\boldsymbol{M}$ coincides with a crystal axis (other than the one taken as the rotation axis); on the other hand, when the field intensity tends to infinity ($1/b = 0$), the magnetization coincides with the field direction.

One of the simplest approaches to get analytical expressions for the magnetic torque would consist in obtaining from (6) a representation of α as a power series in b or $\sin\theta$.

This approach would result in equations valid for both high and low field intensities, instead of expressions valid for large or small θ angles (as those deduced from the Fourier series). The former case presents an interesting alternative, because it would allow one to derive accurate analytical expressions by matching high and low field intensity expansions using a method of analytical continuation. This approach does not seem to have been considered before, even though as we shall show, it is both simple and advantageous.

III. Analytical Expressions for the Magnetic Torque from Power Series

The roots of (6) can be expanded in power series of b and b^{-1}, about $b = 0$ and $1/b = 0$, respectively (Taylor and Laurent series)

$$x = \sum_{n=0}^{\infty} x^{(n)} b^n \tag{8a}$$

$$x = \sum_{n=0}^{\infty} x'^{(n)} b^{-n}. \tag{8b}$$

The coefficients $\{x^{(n)}\}$ and $\{x'^{(n)}\}$ can be easily calculated. In order to keep our approach as simple as possible we will restrict ourselves only to the first few coefficients. Introducing (8a) into (6), and equaling to zero the coefficient premultiplying b^n ($n > 0$), we can derive the coefficients $x^{(n)}$ which are consistent with the condition given by (7) for $b = 0$. Retaining only the terms up to the order b^2, we get

$$x = 1 - (\cos^2\theta/32)b^2 + O(b^3). \tag{9a}$$

Proceeding in a similar way, we introduce (8b) into (6), equal to zero the coefficients of b^{-n} ($n > 0$), and use (7) for the term corresponding to $n = 0$ ($1/b = 0$). Retaining the terms up to the order b^{-2}, the result is

$$\begin{aligned}x = \sin\theta &- 4\left\{\sin\theta - 3\sin^3\theta + 2\sin^5\theta\right\}b^{-1} \\ &+ 16\left\{18\sin^9\theta - 44\sin^7\theta\right. \\ &+ (73/2)\sin^5\theta - (23/2)\sin^3\theta \\ &\left.+ \sin\theta\right\}b^{-2} + O(b^{-3}).\end{aligned} \tag{9b}$$

The matching of expansions (9) can be accomplished through a recently developed method [5]–[7]. This simple procedure has been shown successful at summing different kinds of power series expansions with finite and even zero convergence radii [5]–[7]. As the method takes into account the asymptotic expansions related to the function under study, it is appropriate to use it in this case because the coefficients for both expansions are known.

In order to apply the above-mentioned method it is convenient to define the function $E(\lambda) = x - 1$, where $\lambda = 1/b$. We thus have the following expansions:

$$E(\lambda) = \sum_{n=0}^{\infty} E^{(n)}\lambda^n \qquad (10a)$$

where $E^{(0)} = x'^{(0)} - 1$, and $E^{(n)} = x'^{(n)}$, for $n = 1, 2, \cdots$, about $\lambda = 0$, and

$$E(\lambda) = \lambda^{-2} \sum_{n=0}^{\infty} E'^{(n)}\lambda^{-n} \qquad (10b)$$

where $E'^{(n)} = x^{(n+2)}$, for $n = 0, 1, 2, \cdots$, about $1/\lambda = 0$.

The essential idea in the proposed method (FM [5]–[7]) is to introduce a mapping from the unbounded variable λ (or b) to a new bounded one w, and then to approximate the unknown function $E(\lambda)$ as a sequence of polynomials in the latter variable. The variable w and the sequence of polynomials are determined in order to fulfill two basic conditions: 1) the new approximation to the function must possess the correct analytical structure predicted by both asymptotic expansions (for large and small values of the variable); 2) the polynomials in w must lead to the correct Taylor expansion when rewritten as power series in λ or λ^{-1}. According to the method, the knowledge of the asymptotic expansion for $\lambda >> 1$ allows one to design the most appropriate mapping of the interval $0 \leq \mathrm{Re}\ \lambda < \infty$, *onto* $0 \leq \mathrm{Re}\ w < 1$ [5]–[7]. In our case, as in many other cases, the interval $0 \leq \mathrm{Re}\ \lambda < \infty$ is the only interval relevant for most practical applications. The fact that the new variable is bounded suggests that the poor convergence properties of the power series representation of E in the original variable would be improved in the new representation (i.e., the original divergence would be "smoothed"). In fact, it can be shown rigorously that in most cases the series in powers of w are convergent for $|w| \leq 1$, that is, for all values of interest of the original variable λ [5]–[7].

We will associate to any function having the type of asymptotic expansion characterized by (10b) the following variable [5]–[7]:

$$w = \lambda k/(1 + \lambda k), \qquad k > 0 \qquad (11a)$$

and sequence of polynomials

$$E(\lambda) \approx SE_N = (w/\lambda)^2 S_N(k, w) \qquad (11b)$$

where

$$S_N(k, w) = \sum_{n=0}^{N} e^{(n)} w^n \qquad (11c)$$

with $\{e^{(n)}\}$ a set of k-dependent coefficients to be determined later. The method provides an important degree of freedom in the real parameter k, which can be chosen as shown below. Substituting (11a) into (11b) and (11c), it is clear that the function SE_N possesses the same asymptotic expansions that $E(\lambda)$ for both regimes of the variable λ. This assures us the fulfillment of condition 1) mentioned above. In order to fulfill the second condition, we substitute (11a) in (11b) and (11c), then we expand the expression in power series of λ. Accordingly, we determine the coefficients $\{e^{(n)}\}$ in order to reproduce the original coefficients $\{E^{(n)}\}$. In case of approximating $E(\lambda)$ by SE_N, only the knowledge of the first N coefficients $E^{(n)}$ will be necessary. The final result is simply [5]–[7]

$$e^{(n)} = \sum_{i=0}^{n} (-1)^{n-i} C(-2 - i, n - i) E^{(i)} k^{-2-i} \qquad (12)$$

where $C(a, b) = a(a - 1)(a - 2) \cdots (a - b + 1)/b!$ stands for the combinatorial numbers. The parameter k can be determined as follows: if the sequence $\{SE_N, N = 1, 2, \cdots\}$ is to converge towards $E(\lambda)$ for all λ values for N large enough, then for $1/\lambda \to 0$ ($w \to 1$) the following condition should necessarily be satisfied (cf. (11) and (10b)):

$$\lim_{N\to\infty} \lim_{1/\lambda\to 0} \{S_N(k, w)\} = \lim_{N\to\infty} \{S_N(k, 1)\} = E'^{(0)} \qquad (13)$$

Because the sequence $\{S_N(k, 1)\}$ must converge to a k-independent result (cf. (13)), one can expect that $S_N(k, 1)$, as a function of k, will exhibit a plateau whose extension will increase as N increases. This behavior is qualitatively displayed in Fig. 1. When we compare $S_N(k, 1)$ with $S_{N'}(k, 1)$, $N' > N$, we see that in the latter case the function (11c) shows a more extended and damped oscillation about the exact result. Consequently, a reasonable k value can be obtained if it is chosen within that plateau [5]–[7]. Owing to the fact that the stationary and inflexion points of $S_N(k, 1)$ as a function of k clearly define points belonging to the plateau, they have been chosen as optimum k values [5]–[7]. It is enough for our present purposes to choose k as an inflexion point ($k_N^{(i)}$, from now on); if there is more than one inflexion point, we will choose the one corresponding to the point with lower absolute value of the first derivative.

In the case of $N = 2$, we get the following very simple and useful approximate expression for the function of interest:

$$SE_2 = \{k^* b^{-1}/(1 + k^* b^{-1})\}^2 \{6k^{*-2}E^{(0)} + 4k^{*-3}E^{(1)} + k^{*-4}E^{(2)}\} \qquad (14a)$$

where k^* is different for each θ value, and it is given by

$$k^* = \max k_2^{(i)}, \qquad (k_2^{(i)})_{\pm}^{-1} = (1/20E^{(2)}) \{-24E^{(1)} \pm [576E^{(1)2} - 720E^{(0)}E^{(2)}]^{1/2}\}. \qquad (14b)$$

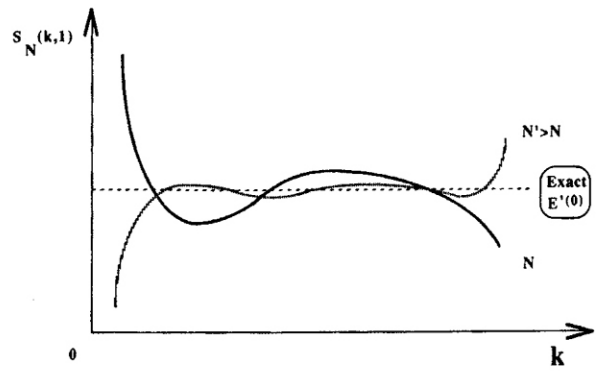

Fig. 1. Qualitative dependence of the sequences $S_N(k, 1)$ on the parameter k, for different values of N (number of Taylor coefficients).

TABLE I
DEPENDENCE OF $\tau' = \tau/K_1$ ON $b = 2HM/K_1$, FOR $\theta = 72°$

b	$-\tau'^{a}$	$-\tau'^{b}$	$-\tau'^{c}$
0.01	0.003086	0.003114	0.003086
0.1	0.030252	0.030559	0.030183
0.5	0.13909	0.14034	0.13796
1.0	0.25138	0.25331	0.24869
2.0	0.41678	0.41912	0.41265
5.0	0.66498	0.66661	0.66293
10.0	0.80439	0.80507	0.80433
100.0	0.93878	0.93878	0.93878

[a]Equations (14a) and (16); $k^* = 6.797809685$.
[b]Equation (14); $k_2^{(i)} = 6.65259310$.
[c]Numerical solution of (5) ("exact" results).

As shown above, the coefficients $E^{(i)}$ are easily deduced from (8)–(10). By using these coefficients, (14) provide an approximate analytic expression for the dimensionless torque τ'

$$\tau' \approx \sin 4\{\arcsin(SE_2 + 1)\} \tag{15}$$

as a function of both b and θ, whose properties we will discuss in the next section.

Another simple expression for τ' can be obtained if k^* is determined in order to satisfy the exact behavior when $b \to 0$. In this case, we simply have to choose k^* so that the following equality is satisfied:

$$S_2(k^*, 1) = E'^{(0)}. \tag{16}$$

As we shall see, results are greatly improved when one takes $E'^{(0)}$ into consideration to perform the calculations. In this case, coefficients for both asymptotic expansions are included explicitly up to the second order, in the correct analytic framework, to reconstruct the function $E(\lambda)$.

It has been shown in [5]–[7] that the procedure discussed above is more advantageous when the series (10a) is sign-oscillating. Consequently, it is expected that there will be an optimum range of values of θ for the application of the present method, corresponding to that in which the sign-oscillation is found. Using (9b) and (10a), it is clear that $E^{(0)} \geq 0$ and $E^{(1)} \leq 0$ in the range of θ values of interest ($45° \leq \theta \leq 90°$). Accordingly, to obtain the oscillation in sign we must determine the range of θ, in the interval above, so that $E^{(2)} \geq 0$. From (9b) it is deduced that

$$63.92° \leq \theta \leq 90°. \tag{17}$$

Thus for $45° \leq \theta \leq 63.92°$ one can expect poorer results when using (14) for $b \to 0$.

IV. Results and Further Comments

We have studied two types of universal curves for the magnetic torque: τ' versus θ (at a fixed b value), and τ' versus b (at a fixed θ angle).

In the first case, we have chosen $\theta = 72°$ (according to (17)) in order to illustrate the quality of the results derived from (14) and (16). Table I shows the results obtained, compared with those deduced from the exact numerical solution of (5). It is clear that the results are highly improved when using (16); nevertheless, both approaches can be considered, because of their simplicity, as excellent analytical approximations to τ' in the whole range of b values.

With regard to the universal curve τ' versus θ, we must choose a physically meaningful value of b. Taking into account the following estimations for the constants [1]–[4]:

$$K_1 \approx 10^5 \text{ erg} \cdot \text{cm}^{-3} \qquad 10^3 < H(\text{Oe}) < 10^5$$
$$0.1 < M(\text{T}) < 10$$

we obtain the range $10 < b < 10^5$. The greater the b value, the better the results; as a consequence, we have chosen $b = 10$ and $b = 100$ as critical examples to test our method. Fig. 2 shows the results obtained using (14) for $45° \leq \theta \leq 90°$. For $b = 100$ the results are indistinguishable from the exact ones (circles); for $b = 10$ the agreement is still excellent. Then we conclude that the method allows one to have an analytic expression for τ' as a function of θ and b, with a precision comparable to that of experimental results.

An expression of τ' as a function of b, instead of H, is more general because b is only proportional to H if M and K_1 are independent of it. This happens only if H is very large.

Let us add a word in regard to the application of the method. Owing to the fact that τ' and b are dimensionless it is necessary, first of all, to determine the scale of the universal curve. This can be simply accomplished by determining the maximum torque ($\tau' = 1$) at a very intense magnetic field, which gives K_1. Using this value and the experimental result for the magnetic torque τ at a fixed angle θ, our method makes it possible to determine the magnetization M.

As we mentioned in Section II, the above analysis is general enough to provide a description of the magnetocrystalline anisotropy effects in different systems, not only in the easy axes of magnetization of a $|001|$ plane in a cubic crystal. To illustrate this point, let us consider first, for example, a group of uniaxial crystals.

For a tetragonal crystal, including only terms up to the

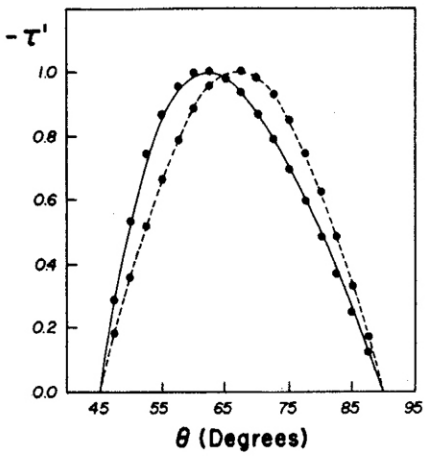

Fig. 2. Dependence of the dimensionless torque $\tau' = \tau/K_1$ on the angle θ for fixed values of $b = 2HM/K_1$. ------: $b = 100$; ——: $b = 10$; ●: exact results.

fourth order, the magnetic anisotropy energy can be written in polar coordinates as [1]

$$E \approx K_1 \sin^2 \alpha + K_2 \sin^4 \alpha + K_3 \sin^4 \alpha \cos 4\varphi \quad (18)$$

with α the angle between the magnetization vector and the crystal axis of the uniaxial system. Analogously, a similar expression can be written for the energy of a uniaxial crystal of hexagonal type [1]

$$E \approx K_1 \sin^2 \alpha + K_2 \sin^4 \alpha + K_3 \sin^6 \alpha + K_4 \sin^6 \alpha \cos 6\varphi. \quad (19)$$

It is interesting to discuss how the present method can be applied to the study of the magnetic properties of the above systems by means of torque experiments. Our study is devoted, as mentioned before, to the design of analytical representations of universal curves useful to the determination of the anisotropy constants. In order to obtain all of them, we need to describe the magnetization under rotation about different axes. This can be accomplished following a strategy that takes advantage of the expressions already deduced in Sections II and III.

If a uniaxial crystal is rotated about a direction perpendicular to the crystal axis, the torque per unit volume is

$$\tau = \partial E/\partial\alpha = (K_1 + K_2) \sin 2\alpha - (1/2) K_2 \sin 4\alpha \quad (20)$$

when a magnetic field $\boldsymbol{H}$, with an angle θ in respect to the crystal axis, is applied. The equilibrium condition becomes in this case

$$(K_1 + K_2) \sin 2\alpha - (1/2) K_2 \sin 4\alpha = HM \sin(\theta - \alpha). \quad (21)$$

It is clear that we will have, once again, an algebraic eighth-degree equation in $x = \sin \alpha$ to describe the torque above. However, even though the structure of the equation will be the same, we have in (21) *two* unknowns in K_1 and K_2. Nevertheless, the problem can be solved in a simple way if one proceeds as follows [1]: First, a composite sample is prepared consisting of two of the above uniaxial crystals (equal in size) with their crystal axes perpendicular to each other. Then, the sample is rotated about a third axis perpendicular to both previous crystal axes. Analyzing the magnetic torque in this condition one obtains

$$\tau = -(1/2) K_2 \sin 4\alpha. \quad (22)$$

This result shows clearly that the anisotropy constant K_2 can be obtained, exactly as earlier discussed for the case of K_1, for the cubic crystal in the $|001|$ plane. After determining K_2, substitution of it into (21) transforms the latter into an equation similar to the one discussed in Section II. Furthermore, if a tetragonal crystal is studied by rotating it about its principal axis, the torque required becomes

$$\tau = -4K_3 \sin 4\alpha. \quad (23)$$

Accordingly, we notice that all anisotropy constants can be derived in a similar fashion for a uniaxial crystal using the same family of universal curves discussed in previous sections.

To study the magnetic properties of cubic crystals out of the $|001|$ plane we must rotate it about a different axis. This will introduce the second anisotropy constant K_2. If the rotation is done, for instance, about the $|0\bar{1}1|$ axis, one obtains [1]

$$\tau = (K_1/4 + K_2/64) \sin 2\alpha + (3K_1/8 + K_2/16) \sin 4\alpha - (3/64) K_2 \sin 6\alpha. \quad (24)$$

Once K_1 is known, the equilibrium condition applied to (24) gives an equation containing K_2 as the only unknown. In this latter case, the presence of a term such as $\sin 6\alpha$ gives rise to an algebraic equation of 12th degree in $\sin \alpha$. The treatment is slightly more complicated in this case, but it can be accomplished following a similar procedure: 1) determine the correlation between the roots in the limits $H \to 0$ and $H \to \infty$; 2) obtain the first few coefficients of the corresponding Taylor and Laurent power series expansions of the roots; 3) use the FM to continuate analytically the series and provide an expression for the new torque as a function of the field intensity and the angle θ. The formulation provided for the FM in Section III is general enough to treat this or any other kind of problems leading to algebraic equations such as (5).

The method explained above throws some light on a unified approach to derive analytical expressions for torque curves, and to provide a strategy for the determination of the anisotropy constants. A combined application of the present procedure for different crystal planes would be also valuable at modeling the low-field portions of hysteresis loops. Work on this possibility is being done at present in our group.

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ELECTROSTATIC AND MAGNETOSTATIC BEHAVIOUR OF ELLIPSOIDAL BODIES

Carlos E.Soliverez

ABSTRACT

A general survey is made of the macroscopic behaviour of ellipsoidal ferroelectric, ferromagnetic, dielectric, paramagnetic, diamagnetic, conducting, and superconducting bodies in uniform electrostatic and magnetostatic applied fields. The determination of the polarisation in saturated ferroelectric and ferromagnetic materials is discussed. In all cases the internal and external fields, energy, force and torque on the body are given in terms of the applied field, of the polarisation, and of the depolarisation tensor $\underset{\sim}{n}$ which completely characterizes the geometry of the problem. The depolarisation tensor is a function only of the ellipsoid's diameters and its principal values inside the body are the familiar demagnetising coefficients. For all non linear materials the polarisation is an implicit function of the applied field, of $\underset{\sim}{n}$, and of the material's properties. In the linear case the polarisation may be expressed as a linear function of the applied field in terms of the polarisability tensor in a way completely analogous to the molecular case. This polarisability tensor is given as a simple function of the body's tensorial susceptibility $\underset{\sim}{\chi}$, and of the interior depolarisation tensor. When taking scalar susceptibilities $\chi = \infty$ and $\chi = -1$ (S.I.units) the case of conductors and superconductors are respectively obtained. The meaning of the fictive polarisations introduced in these two cases is discussed.

"ELECTROSTATIC AND MAGNETOSTATIC BEHAVIOR OF ELLIPSOIDAL BODIES"

C.E. SOLIVEREZ

MANUSCRIPT

LIST OF SYMBOLS

- All symbols underlined with a wavy line are boldface: $\underset{\sim}{M}$
- $\underset{\sim}{1}$ = boldface one
- $\underset{\sim}{O}$, $\underset{\sim}{0}$ = boldface uppercase oh
- All dots between boldface symbols stand for scalar product: $\underset{\sim}{M}.\underset{\sim}{B}$
- ε_0 = epsilon naught (vacuum's electric permeability)
- $\mathfrak{N}$ = german eff
- $\underset{\sim}{\Lambda}$ = boldface upper case lambda
- All crosses between boldface symbols stand for vector product: $\underset{\sim}{A} \times \underset{\sim}{B}$
- $\in$ = "is included in": r $\in$ VV
- $\notin$ = " is not included in": r $\notin$ V
- Δ = sigma
- $\tilde{=}$ = approximately equal to

1. INTRODUCTION

The electrostatic or magnetostatic problem of an arbitrarily shaped body placed in a uniform applied field may in general be solved only in an approximate way. This is a consequence of the non-local character of the problem, where the polarisation in any given point depends on the polarisation of the whole body. Fortunately there is at least one situation, perhaps unique, for which the character of the problem is purely local: the case of ellipsoidal bodies. Here the fields and polarisations may be written in terms of a purely geometrical entity, the depolarisation tensor[1].

Ellipsoids play a distinguished role in electromagnetism because they are the only known finite bodies which when placed in homogeneous applied fields respond with uniform polarisations and internal fields. ~~We should point out here the common error of believing that Maxwell proved the ellipsoidal shapes to be the only ones leading to uniform polarisation and internal field.~~ As will be seen in section 3 this problem may be reduced to the simpler one of seeking the shapes such that a certain integral function V (our $I(\underset{\sim}{r})$) is a second degree polynomial in x, y, z. Quoting Maxwell[2] '...the only cases with which we are acquainted[3] in which V is a quadratic function of the coordinates within the body are those in which the body is bounded by a complete surface of the second degree, and the only case in which such a body is of finite dimensions is when it is an ellipsoid'. It seems reasonable to assume that ellipsoidal bodies are the only finite ones to have the quadratic property, but in our knowledge no one has ever proved this to be so.

italics

From the pedagogical point of view the only shapes discussed at an elementary level are limits of ellipsoids: the semi-infinite slab or thin disc (1 finite diameter, 2 equal and infinite diameters) and the infinitely-long circular cylinder (2 equal and finite diameters, 1 infinite diameter). The analysis made thus is based only on continuity conditions for the fields, leaving the third of the simple shapes, the sphere, for more

advanced treatments. The experimental determination of electric and magnetic polarisations and of magnetic anisotropy constants is usually made by measuring the forces or torques acting on spheres, discs or cylinders placed either in uniform or in carefully controlled inhomogeneous fields[4]. It is of interest in these cases to find the influence of small departures from the perfect shape, departures that may be taken to be of ellipsoidal nature. Conversely, fields may be measured through their action on spheroidal (usually needle-shaped) bodies with known electric and magnetic properties.

From the technological point of view methods such as magnetic separation and electrostatic precipitation, and properties such as the coercivity of magnetic tapes, to cite only a few, depend on the particle's shape distribution, a problem that may be tackled only in the approximation of ellipsoidal shapes[5].

It is therefore baffling to find the problem discussed only in a small number of textbooks on electromagnetism, and that the treatment given in those few textbooks lacks either clarity or generality. In what follows an attempt is made to fill this void by collecting known results that are scattered through the literature, and by generalizing concepts and treatment whenever this was feasible.

All sorts of homogeneous materials will be considered, both isotropic and anisotropic.

The applied field will be taken to have fixed sources, that is to remain unchanged upon the introduction of the body.

For the discussion it is convenient to consider separately three different kinds of materials:

A) ferroelectrics and ferromagnets;
B) dielectrics, diamagnets and paramagnets;
C) conductors and superconductors.

Case A corresponds to materials with a permanent polarisation which does not depend solely on the applied field. In this case the polarisation has to be given as data, although this does not mean that it may be chosen at will, as it is discussed in section 2. For a given uniform polarisation the fields inside and outside the body are a function only of the

depolarisation tensor $\underset{\sim}{n}$. This rank-two symmetric tensor seems to have been previously defined only at interior points[1]. In the literature of magnetism this interior value $\underset{\sim}{N}$ is called (apart from a posible 4π factor) the demagnetisation tensor[6], and its principal values are the demagnetisation factors or coefficients. A simple expression for the shape-anisotropy energy[7], which proves useful for the calculation of torques on these materials, is given here in terms of $\underset{\sim}{N}$.

In case B the polarisation is induced by the applied field. In order to take into account also non linear effects the case is considered where the polarisation is an arbitrary vectorial function $\underset{\sim}{X}$ of the internal macroscopic field. It is then found that for anisotropic ellipsoids both the polarisation and the internal field are homogeneous, being explicitly given, together with the external field, as functions of $\underset{\sim}{n}$, $\underset{\sim}{X}$ and the applied field. This generalizes previous results explicitly given for the internal vectors in the isotropic and linear case, but only suggested for the anisotropic non-linear case[8]. In the linear case a relationship is established between the body's dipolar moment and the applied field through the definition - in a way completely analogous to the molecular case[9] - of a polarisability tensor $\hat{a}$ which is a function of both $\underset{\sim}{N}$ and the susceptibility. The polarisability tensors previously defined for conducting and superconducting ellipsoids[10] are found to be only particular cases of our more general one (see section 4).

Strictly speaking polarisations are either zero or meaningless for the substances belonging to group C. Nevertheless, in this case it is possible to define fictitious polarisations in terms of which the problem is rendered mathematically equivalent to case B. It will be proved here that these fictive polarisations give the right value of the body's dipole moment. It is also found that the polarisability tensor for conducting bodies is obtained from the polarisability tensor for dielectric bodies in the limit of infinite isotropic electric susceptibility, and that in the case of superconductors the correspondence may be established with perfect diamagnets. These results may be taken as a justification, at least from the static point of view, of the frequently drawn analogy between those two sets of substances.

In all three cases formulas are given for the body's energy and for the force and torque experienced within the applied field. Instead of the cumbersome method of separation of variables in ellipsoidal coordinates, the following much simpler algorithm is used for solving cases B and C. First an integral equation is written for the field, equation which contains all boundary conditions. Then a solution drawn in analogy to case A is shown to satisfy the integral equation.

In section 2 the case of permanently polarised materials is discussed, serving as an introduction to the depolarisation tensor and as a guide to the other two more complicated cases. Some general properties of this tensor, as well as explicit expressions for its components are given in section 3. In the two following sections cases B and C are discussed. The problem of ellipsoidal cavities in unbounded media is discussed in section 6, where the ambiguities of this often quoted problem are stated.

Some of the main advantages of the formulation are the simplicity of the final expressions and the independence of any particular choice of coordinate axes. This flexibility is essential for the case of anisotropic materials whose principal axes of susceptibility do not coincide with the ellipsoid's principal axes. It should also be noticed that in the anisotropic case it is not possible to apply the standard method of separation of variables[11].

In all cases V denotes the ellipsoid's volume. The integrals whose limits are not specified are taken over all space, and usually in principal value around the body's surface in order to exclude possible field's discontinuities. In order to render the formulas valid in all systems of units, the previously defined[12] general constants k_1, k_2, k_3, λ, λ', ε_o and μ_o are used.

2. PERMANENTLY POLARISED MATERIALS

In this case, which corresponds to ferroelectric and ferromagnetic materials, the polarisation depends on the body's history (hysteresis) and may be a complicated function of position due to the possible existence of domains. For these reasons the polarisation is not known a priori and it is required its theoretical or experimental determination.

The general theoretical problem is a formidable one which has not yet been completely solved, the reason for this failure being that the phenomenon of hysteresis consists of a large number of discontinuous transitions among metastable states, transitions which are determined by complex balances of energy[13].

From the experimental point of view it is only practical to analize the uniformly polarised case. Some authors apparently believe that this situation may be obtained for any body placed in a sufficiently strong applied field, corresponding to the saturated case where all domains have parallel polarisations. Unfortunately -due to the boundary conditions satisfied by the field on the body's surface this does not always happen to be the case. A necessary condition for obtaining uniform polarisations in strong applied fields is the homogeneity of the macroscopic field inside the body, as was mentioned in the introduction the only case where this is known to be possible is for ellipsoidal bodies.

Let us consider first a uniformly magnetised ferromagnetic ellipsoid in a strong homogeneous applied field $\underset{\sim}{B}_o = \mu_o \underset{\sim}{H}_o$. As there are no conduction currents

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o - \nabla\phi_m(\underset{\sim}{r}),$$

where the scalar magnetic potential ϕ_m due to the magnetisation $\underset{\sim}{M}$ is given by[14]

$$\phi_m(\underset{\sim}{r}) = \frac{\lambda'}{4\pi} \int_V \underset{\sim}{M}.\ \nabla' \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) d^3\underset{\sim}{r}'.$$

In what follows primed and unprimed operators act on $\underset{\sim}{r}'$ and $\underset{\sim}{r}$ respectively. Changing from primed to unprimed operators, for uniform $\underset{\sim}{M}$ one may write

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o + \frac{\lambda'}{4\pi} \nabla\ (\underset{\sim}{M}.\nabla I(\underset{\sim}{r})), \tag{1}$$

where

$$I(\underset{\sim}{r}) = \int_V \frac{d^3\underset{\sim}{r}'}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \tag{2}$$

is a function which depends only on the body's shape. Upon definition of the dimensionless depolarisation tensor $\underset{\sim}{n}(\underset{\sim}{r})$ with components

$$n_{jk}(\underset{\sim}{r}) = -\frac{1}{4\pi} \frac{\partial^2}{\partial x_j \partial x_k} I(\underset{\sim}{r}), \tag{3}$$

it is obtained

$$\nabla(\underset{\sim}{M}.\nabla I(\underset{\sim}{r})) = -4\pi \underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{M}, \tag{4}$$

valid for any constant vector $\underset{\sim}{M}$. Therefore

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o - \lambda' \underset{\sim}{n}(\underset{\sim}{r}).\ \underset{\sim}{M}, \tag{5}$$

and the magnetostatic problem is completely solved whenever $\underset{\sim}{n}(\underset{\sim}{r})$ and $\underset{\sim}{M}$ are known. It should be stressed that equation (5) is expected to hold only for magnetically saturated ellipsoidal bodies where $\underset{\sim}{M}$ is uniform for in homogeneous applied fields.

For points inside the ellipsoid $\underset{\sim}{n}$ is a constant tensor (see section 3), which in what follows will be called the interior depolarisation tensor $\underset{\sim}{N}$. For these interior points the last term in equation (5),

$$\underset{\sim}{H}_{dem} = - \lambda' \underset{\sim}{N} . \underset{\sim}{M} , \qquad (6)$$

is known as the demagnetising field.

At a fixed temperature the magnitude M of the equilibrium magnetisation is usually constant independently of the orientation respect to the crystal axes[15,16] and it is called the spontaneous magnetisation. The orientation of $\underset{\sim}{M}$ is such as to minimize the body's Helmholtz free energy $\tilde{F}_m$[57,17] at constant applied field. Assuming the body to be an assemblage of rigid dipoles -that is neglecting the induced dipolar moments- and using the Lorentz' approximation[18] $\tilde{F}_m$ may be found to be[19]

$$\tilde{F}_m = - \int_V \underset{\sim}{M} . \underset{\sim}{B}_o \, dV - \frac{\mu_o}{2} \int_V \underset{\sim}{M} . \underset{\sim}{H}' \, dV ,$$

where

$$\underset{\sim}{H}' = \underset{\sim}{H}_{dem} + \frac{\lambda'}{3} \underset{\sim}{M} + \lambda' \underset{\sim}{\Lambda} . \underset{\sim}{M}$$

is the dipolar contribution to the local field. The second term in $\underset{\sim}{H}'$ corresponds to the surface magnetisation of the Lorentz sphere, and the third to the field from the dipoles inside this sphere[20]. For uniformly magnetised ellipsoids

$$\tilde{F}_m/V = - \underset{\sim}{M}.\underset{\sim}{B}_o + \frac{1}{2} \lambda' \mu_o \underset{\sim}{M}.\underset{\sim}{N}.\underset{\sim}{M}.$$

$$- \frac{1}{6} \lambda' \mu_o \underset{\sim}{M}.\underset{\sim}{M} - \frac{1}{2} \lambda' \mu_o \underset{\sim}{M}.\underset{\sim}{\Lambda}.\underset{\sim}{M}. \qquad (7)$$

The second term is the shape anisotropy energy[7]; the last two give the lowest order contributions to the magnetocrystalline anisotropy energy, the uniaxial terms[21]. In the saturated isothermal case we are considering, the anisotropy energy is a function only of the orientation of $\underset{\sim}{M}$, hence the name. When considering contributions to the local field of higher order multipoles one obtains aditional terms for the crystalline anisotropy per unit volume A_m, which are generally written from symmetry arguments[22]. A more general expression for $\tilde{F}_m/V$ is thus

$$\tilde{F}_m/V = - \underset{\sim}{M}.\underset{\sim}{B}_o + \frac{1}{2} \lambda' \mu_o \underset{\sim}{M}.\underset{\sim}{N}.\underset{\sim}{M}$$

$$+ A_m(\underset{\sim}{M}). \qquad (8)$$

If $A_m(\underset{\sim}{M})$ and $\underset{\sim}{M}$ were known a priori one could find the orientations of $\underset{\sim}{M}$ which minimize $\tilde{F}_m/V$ as functions of $\underset{\sim}{B}_o$ and $\underset{\sim}{N}$. As it happens, one has first to experimentally determine A_m and $\underset{\sim}{M}$, which is done through torque experiments.

For a torque to be exerted on the body two conditions should be fulfilled. First, $\underset{\sim}{M}$ and $\underset{\sim}{B}$ have to be non-parallel (see equation (9)); second, the energy must depend on the orientation of $\underset{\sim}{M}$ respect to the body. Both conditions reduce to a single one: there should be anisotropy, whether from shape or crystalline origin. To understand the reason for these two conditions let us consider a few examples.

For a spherical ($\underset{\sim}{N} = \frac{1}{3} \underset{\sim}{1}$) and polycrystalline or amorphous body there is no anisotropy[23] and the free energy (8) is a minimum when $\underset{\sim}{M}$ and $\underset{\sim}{B}_o$ are parallel. For any other orientation there is a torque on each dipole, and as $\underset{\sim}{M}$ is uniform

one speaks of a torque acting on $\underset{\sim}{M}$ and tending to align it with the applied field. Nevertheless there is no torque on the body because the energy does not depend on the relative orientation of $\underset{\sim}{M}$ and the body, and the magnetisation may freely rotate respect to the latter as long as there are no dissipative effects.

Let us now consider a non-spherical polycrystalline body with no applied field. As discussed in section 3, for a finite dimensions ellipsoid the principal values of $\underset{\sim}{N}$ are all positive, the smallest one corresponding to the direction of the largest diameter. Therefore in this case the minimum free energy is obtained when $\underset{\sim}{M}$ is directed along the largest dimension. This effect explains the stability of a compass' needle magnetisation. In a similar way the crystalline anisotropy tends to align $\underset{\sim}{M}$ along the directions of minimum energy called directions of easy magnetisation.

If an amorphous ellipsoid is placed in an applied field the energy will be an absolute minimum when the magnetisation is parallel both to the field and the body's largest diameter. Therefore a torque will appear such that tends to rotate the body towards this direction . If the body is kept fixed the energy cannot attain its absolute minimum and $\underset{\sim}{M}$ will take the position that minimizes equation (8) under the imposed constraints.

For these reasons, in the general case when all energy terms are present, one may determine both A_m and M through torque measurements[24]. Once this is done a minimization of equation (8) yields $\underset{\sim}{M}$ for any given applied field and body orientation. Then, as $V\underset{\sim}{M}$ is the ellipsoid's magnetic moment, the torque $\underset{\sim}{\tau}$ acting on the body is[25,26]

$$\underset{\sim}{\tau} = V \underset{\sim}{M} \times \underset{\sim}{B}_o . \qquad (9)$$

In almost uniform fields which vary little inside the body, a force $\underset{\sim}{f}$ is experienced which to a first approximation is given by[26]

$$\underset{\sim}{f} = V (\underset{\sim}{M}.\nabla)\underset{\sim}{B}_o . \qquad (10)$$

The case of homogeneously polarised ferroelectric ellipsoids may be analyzed in a similar fashion. The electrostatic potential $\phi_p(\underset{\sim}{r})$ produced at point $\underset{\sim}{r}$ by a uniform electric polarisation $\underset{\sim}{P}$ is given by[27]

$$\phi_p(\underset{\sim}{r}) = \frac{\lambda}{4\pi\varepsilon_o} \int_V \underset{\sim}{P}.\ \nabla' \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) d^3\underset{\sim}{r}' \ . \qquad (11)$$

The polarisation being established in the presence of a uniform applied field $\underset{\sim}{E}_o$, the total electric field may be written

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o - \nabla\ \phi_p(\underset{\sim}{r})$$

$$= \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o}\ \underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{P}. \qquad (12)$$

In this case

$$\underset{\sim}{E}_{dep} = -\frac{\lambda}{\varepsilon_o}\ \underset{\sim}{N}.\underset{\sim}{P} \qquad (13)$$

is called the depolarisation field.

Assuming rigid electric dipoles, the Helmholtz free energy F_p at constant applied field[28] may be found to be

$$F_p/V = -\ \underset{\sim}{P}.\underset{\sim}{E}_o + \frac{1}{2}\frac{\lambda}{\varepsilon_o}\ \underset{\sim}{P}.\underset{\sim}{N}.\underset{\sim}{P} + A_p(\underset{\sim}{P}), \qquad (14)$$

where A_p seems to have no specific name in the ferroelectr case[29]. The electric torque $\underset{\sim}{\tau}$ is[30,31]

$$\underset{\sim}{\tau} = V\underset{\sim}{P} \times \underset{\sim}{E}_o, \qquad (15)$$

and the force $\underset{\sim}{f}$ experienced in an almost uniform field $\underset{\sim}{E}_o$ is[31]

$$\underset{\sim}{f} = V(\underset{\sim}{P}.\nabla)\ \underset{\sim}{E}_o \ . \qquad (16)$$

3. PROPERTIES OF THE DEPOLARISATION TENSOR

By definition the depolarisation tensor components $n_{jk}(\underset{\sim}{r})$ are given by

$$n_{jk}(\underset{\sim}{r}) = -\frac{1}{4\pi}\frac{\partial^2}{\partial x_j \partial x_k} I(\underset{\sim}{r}), \tag{3}$$

where

$$I(\underset{\sim}{r}) = \int_V \frac{d^3\underset{\sim}{r}'}{|\underset{\sim}{r}-\underset{\sim}{r}'|} . \tag{2}$$

Equation (3) may also be written as an integral over the body's surface S. That is

$$n_{jk}(\underset{\sim}{r}) = \frac{1}{4\pi}\frac{\partial}{\partial x_j}\int_V \frac{\partial}{\partial x'_k}\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) d^3\underset{\sim}{r}'$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial x_j}\int_S \frac{dS_k}{|\underset{\sim}{r}-\underset{\sim}{r}'|} , \tag{17}$$

where use has been made of the theorem of the gradient[32], and dS_k is the surface element normal to x_k.

According to equations (3) and (2) it is seen that the depolarisation tensor may be derived from the electrostatic potential $I(\underset{\sim}{r})$ of a uniform charge density $\rho = 1/k_1$ distributed over volume V, or from the corresponding gravitational potential[33]. Such a derivation offers the great advantage of putting at our disposal the whole artillery of potential theory, ~~a method~~ the results of which we will ~~make~~ be intensive use of in this section.

It is well known that both the improper integral $I(\underset{\sim}{r})$ and its first derivatives exist and are continuous throughout

all space[34]. It may also be shown that its second derivatives exist everywhere except on the body's surface S, and that the order in which one takes these derivatives may be freely interchanged[35]. It then follows that the depolarisation tensor is symmetric,

$$n_{jk}(\underset{\sim}{r}) = n_{kj}(\underset{\sim}{r}). \tag{18}$$

The most important consequence of this property is that for every point $\underset{\sim}{r}$ there is a set of three orthogonal axes referred to which $\underset{\sim}{n}$ becomes diagonalised. One should bear in mind that this set of principal axes is not necessarily the same for all points.

One of the most useful properties of $\underset{\sim}{n}$ concerns the value of its trace,

$$\mathrm{Tr}\ \{\underset{\sim}{n}\} = \sum_j n_{jj} = -\frac{1}{4\pi}\nabla^2 I(\underset{\sim}{r})$$

$$= -\frac{1}{4\pi}\int_V \nabla^2\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) d^3\underset{\sim}{r}' = \begin{cases} 1 & \text{if } \underset{\sim}{r} \in V, \\ 0 & \text{if } \underset{\sim}{r} \notin V, \end{cases} \tag{19}$$

according to the familiar relationship[36]

$$\nabla^2\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) = -4\pi\,\delta(\underset{\sim}{r}-\underset{\sim}{r}'), \tag{20}$$

where $\delta(\underset{\sim}{r}-\underset{\sim}{r}')$ is Dirac's delta distribution. Equation (19) clearly shows that $\underset{\sim}{n}$ has a discontinuity of the first

kind on the surface S. It should be realized that equation (19) is nothing but Gauss' law.

It may be proved that the interior depolarisation tensor $\underset{\sim}{N}$ is always nonnegative; moreover it is positive definite for bodies of finite dimensions[6]. From equation (19), outside the body

$$\mathrm{Tr}\ \{\underset{\sim}{n}(\underset{\sim}{r})\} = \sum_j n_j(\underset{\sim}{r}) = 0\,,$$

where n_j are the principal values (eigenvalues) of $\underset{\sim}{n}$. Therefore either all the eigenvalues vanish, in which case $\underset{\sim}{n} = 0$ at that point, or the eigenvalues do not all have the same sign. It then follows that the exterior depolarisation tensor is an indefinite tensor[37]

The discontinuity of $\underset{\sim}{n}$ across the surface will now be analyzed using the aforementioned electrostatic analogy. If one writes

$$\phi_f(\underset{\sim}{r}) = -k_1\, \underset{\sim}{O}.\ \nabla I(\underset{\sim}{r})$$

$$= k_1 \int_V \underset{\sim}{O}.\ \nabla' \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) d^3\underset{\sim}{r}'\,,$$

where $\underset{\sim}{O}$ is an arbitrary constant vector, it follows from equation (11) that ϕ_f is the generated electrostatic potential when volume V has a uniform polarisation $\underset{\sim}{O}$[38]. At the same time the polarisation $\underset{\sim}{O}$ is equivalent to a surface (density charge) $\sigma(\underset{\sim}{r})$. Taking into account that

$$\underset{\sim}{O}.\ \nabla' \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) = \nabla' \cdot \left(\frac{\underset{\sim}{O}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\,,$$

replacing in the previous formula and using the theorem of Gauss-Ostrogradsky it is obtained

$$\phi_f(\underset{\sim}{r}) = - k_1 \underset{\sim}{0}. \nabla I(\underset{\sim}{r}) = k_1 \int_S \frac{\underset{\sim}{0}.d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} . \qquad (21)$$

Therefore ϕ_f is also the electrostatic potential due to a surface charge density

$$\sigma(\underset{\sim}{r}) = \underset{\sim}{s}(\underset{\sim}{r}). \underset{\sim}{0} \qquad (22)$$

distributed on the boundary surface S, where $\underset{\sim}{s}(\underset{\sim}{r})$ is the unit normal vector of S at the point $\underset{\sim}{r}$. The associated electric field is

$$\underset{\sim}{E}_f(\underset{\sim}{r}) = - \nabla\phi_f(\underset{\sim}{r}) = k_1 \nabla (\underset{\sim}{0}. \nabla I(\underset{\sim}{r}))$$

$$= - 4\pi k_1 \underset{\sim}{n}(\underset{\sim}{r}). \underset{\sim}{0}, \qquad (23)$$

in accordance with equation (4). From electrostatic theory[39] it is known that $\underset{\sim}{E}_f$ is continuous everywhere except on S where its normal component has the discontinuity of the first kind

$$\underset{\sim}{s}. (\underset{\sim}{E}_f^+ - \underset{\sim}{E}_f^-) = 4\pi k_1 \sigma , \qquad (24)$$

the upper + and - signs referring respectively to the outside and inside of surface S. From equations (22) and (23) it follows that

$$\underset{\sim}{s}.(\underset{\sim}{E}_f^+ - \underset{\sim}{E}_f^-) = 4\pi k_1 \underset{\sim}{s}. (\underset{\sim}{n}^- - \underset{\sim}{n}^+). \underset{\sim}{0}$$

$$= 4\pi k_1 \underset{\sim}{s}.\underset{\sim}{0}.$$

As $\underset{\sim}{0}$ is an arbitrary vector and $\underset{\sim}{n}$ is a symmetric tensor, it is obtained

$$\underset{\sim}{s}.\ (\underset{\sim}{n}^{-} - \underset{\sim}{n}^{+}) = (\underset{\sim}{n}^{-} - \underset{\sim}{n}^{+}).\underset{\sim}{s} = \underset{\sim}{s} \tag{25}$$

for any point on the boundary surface S. As the field's components tangential to S are continuous across the boundary, then

$$\underset{\sim}{t}(\underset{\sim}{r}).(\underset{\sim}{E}_f^{+}(\underset{\sim}{r}) - \underset{\sim}{E}_f^{-}(\underset{\sim}{r})) = 0,$$

where $\underset{\sim}{t}$ is any unit vector tangent to S at point $\underset{\sim}{r}$. It is then easily shown that

$$\underset{\sim}{t}.(\underset{\sim}{n}^{-} - \underset{\sim}{n}^{+}) = (\underset{\sim}{n}^{-} - \underset{\sim}{n}^{+}).\underset{\sim}{t} = 0. \tag{26}$$

As any vector $\underset{\sim}{A}$ defined on S may be written as the sum of two vectors, $\underset{\sim}{A}_s$ normal to the surface and $\underset{\sim}{A}_t$ tangential to it,

$$\underset{\sim}{A} = \underset{\sim}{A}_s + \underset{\sim}{A}_t = \underset{\sim}{s}\,\widehat{\ }\,(\underset{\sim}{s}.\underset{\sim}{A}) - \underset{\sim}{s} \times (\underset{\sim}{s}\times\underset{\sim}{A}),$$

it follows that

$$\underset{\sim}{A}_s = (\underset{\sim}{n}^{-} - \underset{\sim}{n}^{+}).\ \underset{\sim}{A}, \tag{27}$$

$$\underset{\sim}{A}_t = \underset{\sim}{A} - \underset{\sim}{A}_s = (\underset{\sim}{1} + \underset{\sim}{n}^{+} - \underset{\sim}{n}^{-}).\ \underset{\sim}{A}. \tag{28}$$

According to the first Maxwell equation,

$$\nabla\ .\ \underset{\sim}{E}_f(\underset{\sim}{r}) = 4\pi k_1\ \rho(\underset{\sim}{r}),$$

where $\rho(\underset{\sim}{r})$ is the electric charge density, zero everywhere except on the surface S where it is a distribution. Therefore according to equation (23)

$$\nabla\ .\ (\underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{0}) = 0 \tag{29}$$

for every point $\underset{\sim}{r}$ not on the boundary surface S.

Only far apart from the body it is possible to write a general expression for $\underset{\sim}{n}$. This comes about because at large distances the fictitious field $\underset{\sim}{E}_f$ equation (23) may be approximated by the field of a point dipole with electric dipole moment

$$\underset{\sim}{p} = V\underset{\sim}{0}.$$

According to equation (12) and in this approximation it follows that

$$\underset{\sim}{E}_f(\underset{\sim}{r}) = -4\pi k_1 \underset{\sim}{n}(\underset{\sim}{r}).\ \underset{\sim}{0} \cong k_1 \left(\frac{3\ \underset{\sim}{r}(\underset{\sim}{r}.\underset{\sim}{p})}{r^5} - \frac{\underset{\sim}{p}}{r^3} \right),$$

where the origin of coordinates should be taken to be inside the body. One finds that then far away from the body

$$\underset{\sim}{n}(\underset{\sim}{r}) \cong \frac{V}{4\pi} \frac{\underset{\sim}{1} - 3\ \underset{\sim}{r}_o\underset{\sim}{r}_o}{r^3}, \qquad (30)$$

where $\underset{\sim}{r}_o$ is the unit vector in the direction of $\underset{\sim}{r}$, $\underset{\sim}{1}$ the unit dyadic, and $\underset{\sim}{r}_o\underset{\sim}{r}_o$ a dyadic operator[40].

When the interior depolarisation tensor $\underset{\sim}{n}^-$ is known, $\underset{\sim}{n}^+$ may be explicitly written from equations (25) and (26), reading

$$\underset{\sim}{n}^+(\underset{\sim}{r}) = \underset{\sim}{n}^-(\underset{\sim}{r}) - \underset{\sim}{s}(\underset{\sim}{r})\underset{\sim}{s}(\underset{\sim}{r}). \qquad (31)$$

Equation (31) also provides a useful approximation for near external points.

Later on use will be made of the expression

(eq. (67))

$$\underset{\sim}{B} = \nabla\psi = \nabla \left(\nabla \cdot \int' \frac{\underset{\sim}{n}(\underset{\sim}{r}') \cdot \underset{\sim}{0}}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d^3\underset{\sim}{r}' \right) ,$$

where $\underset{\sim}{0}$ is an arbitrary constant vector, and the integration is over all space excluding the surface S. In order to evaluate $\underset{\sim}{B}$ we first set

$$\underset{\sim}{G}(\underset{\sim}{r}') = \underset{\sim}{n}(\underset{\sim}{r}') \cdot \underset{\sim}{0}.$$

Taking into account that

$$\nabla \cdot \left(\frac{\underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) = \nabla \left(\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) \cdot \underset{\sim}{G}(\underset{\sim}{r}') = -\nabla' \left(\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) \cdot \underset{\sim}{G}(\underset{\sim}{r}'),$$

and that according to equation (29)

$$\nabla' \cdot \left(\frac{\underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) = \nabla' \left(\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) \cdot \underset{\sim}{G}(\underset{\sim}{r}') + \frac{\nabla' \cdot \underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$$

$$= \nabla' \left(\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) \cdot \underset{\sim}{G}(\underset{\sim}{r}'),$$

for every point $\underset{\sim}{r}'$ not on S, it follows that

$$\psi = \nabla \cdot \int' \frac{\underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d^3\underset{\sim}{r}' = - \int' \nabla' \cdot \left(\frac{\underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) d^3\underset{\sim}{r}'.$$

Upon application of the theorem of Gauss-Ostrogradsky separately in the two regions determined by S, and after addition, it is obtained

$$\int' \nabla' \cdot \left(\frac{\underset{\sim}{G}(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) d^3\underset{\sim}{r}' = \int_{S_\infty} \frac{\underset{\sim}{G}(\underset{\sim}{r}').d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}$$

$$- \int_S \frac{(\underset{\sim}{G}^+(\underset{\sim}{r}')-\underset{\sim}{G}^-(\underset{\sim}{r}')).d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \quad .$$

From equation (30) it is seen that the integral over the surface at infinity S_∞ vanishes. It is then found from equation (25) that

$$\psi = \int_S \frac{\underset{\sim}{s}.(\underset{\sim}{n}^+(\underset{\sim}{r}')- \underset{\sim}{n}^-(\underset{\sim}{r}')).\underset{\sim}{0}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \, dS$$

$$= - \underset{\sim}{0}. \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \quad .$$

Therefore

$$\underset{\sim}{B} = \nabla\psi = - \nabla\left(\underset{\sim}{0}. \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right)$$

$$= - (\underset{\sim}{0}.\nabla) \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} - \underset{\sim}{0} \times \left(\nabla \times \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right), \qquad (32)$$

where in the last term use has been made of a general property of the nabla operator[41]. Each term in equation

(32) will now be analyzed separately. From the theorem of the rotational[32] and $\nabla\times\nabla g = 0$ for all g, it follows that

$$\nabla\times\int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = \int_S \nabla\times\left(\frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) = \int_S \nabla\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\times d\underset{\sim}{S}$$

$$= \int_S d\underset{\sim}{S}\times\nabla'\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) = \int_V \nabla'\times\left(\nabla'\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\right) d^3\underset{\sim}{r}' = 0.$$

Equation (32) may then be written

$$\underset{\sim}{B} = -(\underset{\sim}{0}.\nabla)\int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = -(\underset{\sim}{0}.\nabla)\sum_j \underset{\sim}{e}_j \int_S \frac{dS_j}{|\underset{\sim}{r}-\underset{\sim}{r}'|}$$

$$= -\sum_j \underset{\sim}{e}_j \int_S \underset{\sim}{0}.\nabla\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right) dS_j = -\int_S d\underset{\sim}{S}\left(\underset{\sim}{0}.\nabla\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\right)$$

$$= -\int_V \nabla'\left(\underset{\sim}{0}.\nabla\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\right) d^3\underset{\sim}{r}' = \int_V \nabla\left(\underset{\sim}{0}.\nabla\left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\right)\right) d^3\underset{\sim}{r}',$$

$$= \nabla\,(\underset{\sim}{0}.\nabla I(\underset{\sim}{r})),$$

where use has been made of the theorem of the gradient[32]. From equation (4) it is finally obtained

$$\underset{\sim}{B} = \nabla\left(\nabla\cdot\int'\frac{\underset{\sim}{n}(\underset{\sim}{r}').\underset{\sim}{0}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\, d^3\underset{\sim}{r}'\right)$$

$$= \nabla\int_S \frac{d\underset{\sim}{S}.(\underset{\sim}{n}^+(\underset{\sim}{r}')-\underset{\sim}{n}^-(\underset{\sim}{r}')).\underset{\sim}{0}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}$$

$$= -4\pi\,\underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{0}. \qquad (33)$$

The previously discussed properties are completely general ones, valid for arbitrary shapes. In the remaining of the section the case is considered where V is the volume inside the ellipsoidal surface

$$(x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1 ,$$

the a_j's being the ellipsoid's semi-axes. Solving the electrostatic potential $I(\underset{\sim}{r})$ of a uniform charge density $\rho = 1/k_1$ distributed over V it is found[42,43]

$$I(\underset{\sim}{r}) = \pi a_1 a_2 a_3 \int_\kappa^\infty ds \left(1 - \sum_{j=1}^{3} x_j^2/R_j^2(s)\right)/R_1(s)R_2(s)R_3(s),$$

where $$R_j(s) = (s + a_j^2)^{1/2} ,$$

the x_j's being the coordinates of the observation point $\underset{\sim}{r}$. For interior points it is

$$\kappa = 0 ,$$

and for exterior points $\kappa(\underset{\sim}{r})$ is the largest algebraic root of the third-degree equation

$$1 - \sum_{j=1}^{3} x_j^2/R_j^2(\kappa) = 0. \qquad (34)$$

It then follows from equations (3) and (34) that

$$n_{jk}(\underset{\sim}{r}) = \frac{1}{2}\, a_1 a_2 a_3\, \delta_{jk} \int_{\kappa}^{\infty} \frac{ds}{R_j^{\,2}(s) R_1(s) R_2(s) R_3(s)}$$

$$- \frac{1}{2}\, a_1 a_2 a_3 \quad \frac{x_j}{R_j^{\,2}(\kappa) R_1(\kappa) R_2(\kappa) R_3(\kappa)} \quad \frac{\partial \kappa}{\partial x_k}$$

where δ_{jk} is the Kronecker's delta. Notice that for interior points

$$\frac{\partial \kappa}{\partial x_k} = 0 \ ,$$

whilst for exterior points it is obtained from equation (34)

$$\frac{\partial \kappa}{\partial x_k} = 2x_k / R_k^{\,2}(\kappa) \Big(\sum_m x_m^{\,2} / R_m^{\,4}(\kappa) \Big) .$$

Upon definition of

$$\Delta(\underset{\sim}{r}) = \begin{cases} 0 & \text{if } \underset{\sim}{r} \in V \ , \\ 1 & \text{if } \underset{\sim}{r} \notin V \ , \end{cases}$$

one way write

$$n_{jk}(\underset{\sim}{r}) = \delta_{jk}\, \frac{1}{2}\, a_1 a_2 a_3 \int_{\kappa}^{\infty} \frac{ds}{R_j^{\,2}(s) R_1(s) R_2(s) R_3(s)}$$

$$-\Delta(\underset{\sim}{r})\ a_1 a_2 a_3 x_j x_k / R_1(\kappa) R_2(\kappa) R_3(\kappa) R_j^{\,2}(\kappa) R_k^{\,2}(\kappa) \left(\sum_m x_m^{\,2} / R_m^{\,4}(\kappa)\right). \quad (35)$$

From equation (35) it is easily seen that the interior depolarisation tensor $\underset{\sim}{N}$ is constant, and when referred to the ellipsoid's system of principal axes it is also diagonal. These properties do not hold for the exterior tensor, as may be seen from instance from the dipolar character of the fields far away from the body. It also follows that $\underset{\sim}{N}$ does not depend on the ellipsoid's volume, being a function only of the relative values of any two semi-axes respect to the third. Therefore two ellipsoids with semi-axes a_1, a_2, a_3 and ca_1, ca_2, ca_3, where c is any real positive number, have exactly the same $\underset{\sim}{N}$ (but, because of κ, they do not have the same exterior depolarisation tensor). Some important consequences of this last property will be discussed in section 6.

When the lengths of all semi-axes are different the integrals in equations (35) are elliptic integrals which for the purposes of numerical calculations are conveniently expressed in terms of the normal incomplete elliptic integrals of the first and second kind[44]. The corresponding expressions for $\underset{\sim}{N}$ have been given in the literature[45,46] for this general case as well as for spheroids and other limiting cases where the integrals are expressible in terms of elementary functions. Expressions for the exterior depolarisation tensor may be derived from $\underset{\sim}{I}(\underset{\sim}{r})$[43] in a similar fashion and will not be given here. The case of the sphere, thin disc and infinitely-long circular cylinder are most easily solved through recourse to Gauss' theorem and provide very illuminating examples of the formalism.

The principal values of $\underset{\sim}{N}$ -the demagnetisation coefficients N_j- should not be mistaken with the ballistic or magnetometric demagnetisation factors which are defined for non-ellipsoidal

bodies[47,48] as suitable averages of the there position-dependent tensor $\underset{\sim}{N}$.

From the tabulated values it is seen that when the principal axes are succesively numbered in order of increasing a_j's, the ordered N_j's form a decreasing sequence. The smallest N_j tends to zero when the relative length of the largest semi-axis tends to infinity and the largest N_j tends to unity when the relative length of the smallest semi-axis tends to zero.

As $\underset{\sim}{N}$ is a constant tensor (for ellipsoids) it remains invariant under a change of coordinates which is a symmetry operation for the body. Therefore $\underset{\sim}{N}$ obeys the known symmetry properties of rank-two tensors[49,6]. Thus, if any two axes are equivalent the corresponding N_j's are equal. This, together with the trace rule equation (19) gives for the sphere

$$\underset{\sim}{N}_{\text{sphere}} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

For an infinitely-long circular cylinder with $a_3 = \infty$ (or alternatively $a_1 = a_2 = 0$, $a_3 \neq 0$) it is $N_3 = 0$ and therefore

$$\underset{\sim}{N}_{\text{cylinder}} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In a similar fashion it is found that for the thin disc $a_1 = a_2$, $a_3 = 0$ (or alternatively $a_1 = a_2 = \infty$, $a_3 \neq 0$)

$$\underset{\sim}{N}_{disc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

4. INDUCED POLARISATION

This is the case of dielectric, diamagnetic and paramagnetic materials, where the polarisation at any point is a single-valued function of the macroscopic field at that same point.

For the dielectric case it may be written[27]

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o + \frac{\lambda}{4\pi\varepsilon_o} \nabla \int_V \underset{\sim}{P}(\underset{\sim}{r}') . \nabla \left(\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \right) d^3\underset{\sim}{r}' , \qquad (36)$$

where $\underset{\sim}{E}_o$ is the uniform applied field and $\underset{\sim}{P}$ is the induced electric polarisation which cannot be assumed a priori to be uniform. Actually in this case $\underset{\sim}{P}$ is homogeneous, but we will have to prove it.

For maximum generality $\underset{\sim}{P}$ will be taken to be a vectorial function of $\underset{\sim}{E}$,

$$\underset{\sim}{P}(\underset{\sim}{r}) = \varepsilon_o \underset{\sim}{\chi}_e (\underset{\sim}{E}(\underset{\sim}{r})), \qquad (37)$$

and as only homogeneous bodies are considered the dependence of $\underset{\sim}{P}$ on $\underset{\sim}{r}$ comes only through that of $\underset{\sim}{E}$. For sufficiently small fields the linear relationship

$$\underset{\sim}{P}(\underset{\sim}{r}) = \varepsilon_o \underset{\sim}{\chi}_e \cdot \underset{\sim}{E}(\underset{\sim}{r}) \qquad (38)$$

holds, where $\underset{\sim}{\chi}_e$ is the constant[50] electric susceptibility tensor. Through a simple mathematical manipulation of equation (36) and from equation (37), it is found

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o + \frac{\lambda}{4\pi} \nabla \left(\nabla . \int_V \frac{\underset{\sim}{\chi}_e(\underset{\sim}{E}(\underset{\sim}{r}'))}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d^3\underset{\sim}{r}' \right) . \qquad (39)$$

This is the sought after integral equation satisfied by $\underset{\sim}{E}$, equation which already contains all boundary conditions. It is the integration over $\underset{\sim}{r}'$ which gives the problem its non-local character. Equation (39) may be looked upon as a set of three coupled nonhomogeneous Fredholm equations of the second kind[51], which may be solved by succesive approximations[52]. As the solution of equation (39) must be unique it is much more simpler to take an Ansätz in analogy to equation (12). Therefore we try

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{n}(\underset{\sim}{r}) . \underset{\sim}{Q}, \qquad (40)$$

where $\underset{\sim}{Q}$ is for the time being a constant vector, devoid of any physical meaning, which should be chosen in such a way that equation (39) is satisfied. From equations (39) and (40) it is found

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o + \frac{\lambda}{4\pi} \nabla \left(\nabla . \int_V \frac{\underset{\sim}{\chi}_e(\underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N}.\underset{\sim}{Q})}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, d^3\underset{\sim}{r}' \right)$$

$$= \underset{\sim}{E}_o + \frac{\lambda}{4\pi} \nabla \left(\underset{\sim}{X}_e (\underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N}.\underset{\sim}{Q}) . \nabla I(\underset{\sim}{r}) \right)$$

$$= \underset{\sim}{E}_o - \lambda \underset{\sim}{n}(\underset{\sim}{r}) . \underset{\sim}{X}_e (\underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N}.\underset{\sim}{Q}) , \qquad (41)$$

where in the last step use has been made of equation (4). Notice that the result is only valid if $\underset{\sim}{N}$ is not a function of position as it happens for ellipsoids; otherwise $\underset{\sim}{X}_e$ may not be taken outside the integral. From equations (40) and (41) it follows that

$$\underset{\sim}{Q} = \varepsilon_o \underset{\sim}{X}_e (\underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N}.\underset{\sim}{Q}) , \qquad (42)$$

which when solved gives the constant vector $\underset{\sim}{Q}$ to be used in the solution equation (40). It is thus seen that the field inside a dielectric ellipsoid placed in a uniform applied field is uniform, and that the dependence on $\underset{\sim}{r}$ of the field outside the ellipsoid comes only from $\underset{\sim}{n}(\underset{\sim}{r})$. Comparing equations (42) and (37), and taking into account equation (40), it is seen that $\underset{\sim}{Q}$ is nothing but the ellipsoid's uniform electric polarisation $\underset{\sim}{P}$. Summing up

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{n}(\underset{\sim}{r}) . \underset{\sim}{P}, \qquad (43\ a)$$

where

$$\underset{\sim}{P} = \varepsilon_o \underset{\sim}{X}_e (\underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N} . \underset{\sim}{P}) \quad . \qquad (43\ b)$$

For interior points equation (43a) corresponds to an equation previously given by Landau and Lifshitz with the cryptic remark[53] that '*the existence of such a relation follows from the form of the boundary conditions...*'. (*italics*)

For the linear case equation (38) $\underset{\sim}{P}$ and the body's electric dipole moment $\underset{\sim}{p}$ may be explicitly written

$$\underset{\sim}{P} = \frac{\varepsilon_o}{V} \, \underset{\sim}{\alpha}_e \cdot \underset{\sim}{E}_o \, , \tag{44}$$

$$\underset{\sim}{p} = V\underset{\sim}{P} = \varepsilon_o \, \underset{\sim}{\alpha}_e \cdot \underset{\sim}{E}_o \, , \tag{45}$$

where

$$\underset{\sim}{\alpha}_e = V \, (\underset{\sim}{1} + \lambda \, \underset{\sim}{\chi}_e \cdot \underset{\sim}{N})^{-1} \cdot \underset{\sim}{\chi}_e = V \, (\underset{\sim}{\chi}_e^{-1} + \lambda \, \underset{\sim}{N})^{-1} \tag{46}$$

is the body's electric polarisability tensor. This symmetric rank-two tensor is defined in analogy to the molecular case[9], and has been previously discussed for conducting ellipsoids[10].

In the general case equation (37) the following expression holds for the polarisation energy[54]

$$F_p = - V \int_0^{\underset{\sim}{E}_o} \underset{\sim}{P}(\underset{\sim}{E}_o) \cdot d\underset{\sim}{E}_o . \tag{47}$$

For the linear case equation (45), equations (47), (15) and (16) may be written[54,30,31]

$$F_p = -\frac{1}{2} \quad \varepsilon_o \underset{\sim}{E}_o \cdot \underset{\sim}{\alpha}_e \cdot \underset{\sim}{E}_o \ , \tag{48}$$

$$\underset{\sim}{\tau} = \varepsilon_o \ (\underset{\sim}{\alpha}_e \cdot \underset{\sim}{E}_o) \times \underset{\sim}{E}_o \ , \tag{49}$$

$$\underset{\sim}{f} = \varepsilon_o \ \nabla \ (\underset{\sim}{E}_o \cdot \underset{\sim}{\alpha}_e \cdot \underset{\sim}{E}_o) \ . \tag{50}$$

Expressions (48) and (49) have been previously discussed by Stratton for the isotropic case[55], where the stable equilibrium for a freely rotating body is achieved when the ellipsoid's largest semi-axis is parallel to the applied field. For the anisotropic case the corresponding direction is that of the maximum principal value of $\underset{\sim}{\alpha}_e$, where $\underset{\sim}{\tau}$ is zero and F_p a minimum. The reader should compare the complexity of the equations obtained by Stratton in the body's system of principal axes with the coordinates-independent equations given here.

For the diamagnetic and paramagnetic case it may be written[14,76]

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o + \frac{\lambda'}{4\pi} \ \nabla \int_V \underset{\sim}{M}(\underset{\sim}{r}') \cdot \nabla \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) d^3\underset{\sim}{r}' \ , \tag{51}$$

$$\underset{\sim}{M}(\underset{\sim}{r}) = \underset{\sim}{X}_m (\underset{\sim}{H}(\underset{\sim}{r})) \ . \tag{52}$$

In the same fashion as in the dielectric case $\underset{\sim}{H}$ is found to obey the integro-differential equation

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o + \frac{\lambda'}{4\pi} \nabla \left(\nabla . \int_V \frac{\underset{\sim}{X}_m(\underset{\sim}{H}(\underset{\sim}{r}'))}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \, d^3\underset{\sim}{r}' \right) , \qquad (53)$$

its solution being

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{H}_o - \lambda' \, \underset{\sim}{n}(\underset{\sim}{r}) . \underset{\sim}{M}, \qquad (54)$$

where

$$\underset{\sim}{M} = \underset{\sim}{X}_m (\underset{\sim}{H}_o - \lambda' \, \underset{\sim}{N} . \underset{\sim}{M}) \qquad (55)$$

is the uniform body's magnetisation.

It should be noticed that equation (55) is the correct starting point in the molecular field approximation for a single-domain ferromagnet[56]. In the isotropic case $\underset{\sim}{X}_m$ is a scalar function which is usually taken to be -apart from a constant factor- the Brillouin function.

For the linear case, where equation (52) may be written

$$\underset{\sim}{M}(\underset{\sim}{r}) = \underset{\sim}{\chi}_m . \underset{\sim}{H}(\underset{\sim}{r}) , \qquad (56)$$

where $\underset{\sim}{\chi}_m$ is the constant magnetic susceptibility tensor, equation (55) may be solved giving

$$\underset{\sim}{M} = \frac{1}{V} \, \underset{\sim}{\alpha}_m . \underset{\sim}{H}_o , \qquad (57)$$

$$\underset{\sim}{m} = V\underset{\sim}{M} = \underset{\sim}{\alpha}_m . \underset{\sim}{H}_o . \qquad (58)$$

$\underset{\sim}{m}$ is the body's magnetic dipole moment, and $\underset{\sim}{\alpha}_m$ the symmetric body's magnetic polarisability tensor

$$\underset{\sim}{\alpha}_m = V(\underset{\sim}{1} + \lambda' \underset{\sim}{\chi}_m \cdot \underset{\sim}{N})^{-1} \cdot \underset{\sim}{\chi}_m = V(\underset{\sim}{\chi}_m^{-1} + \lambda' \underset{\sim}{N})^{-1} , \qquad (59)$$

previously defined for superconducting ellipsoids[10]. The so-called shape effects come about because $\underset{\sim}{\alpha}_m \neq V\underset{\sim}{\chi}_m$. One may disregard these effects in the diamagnetic case where $\lambda' \underset{\sim}{\chi}_m \sim 10^{-5}$, but not always in the paramagnetic case -as erroneously stated by some authors- where the contribution may be significative. The relevant expression for the magnetisation energy $\tilde{F}_m$ is[57]

$$\tilde{F}_m = -\mu_o V \int_0^{\underset{\sim}{H}_o} \underset{\sim}{M}(\underset{\sim}{H}_o) \cdot d\underset{\sim}{H}_o . \qquad (60)$$

For the linear case $\tilde{F}_m$, the torque and the force equations (9) and (10) may be written[57,25,26]

$$\tilde{F}_m = -\frac{\mu_o}{2} \underset{\sim}{H}_o \cdot \underset{\sim}{\alpha}_m \cdot \underset{\sim}{H}_o , \qquad (61)$$

$$\underset{\sim}{\tau} = \mu_o (\underset{\sim}{\alpha}_m \cdot \underset{\sim}{H}_o) \times \underset{\sim}{H}_o , \qquad (62)$$

$$\underset{\sim}{f} = \mu_o \nabla (\underset{\sim}{H}_o \cdot \underset{\sim}{\alpha}_m \cdot \underset{\sim}{H}_o) . \qquad (63)$$

At constant temperature stable equilibrium is achieved when $\tilde{F}_m$ is a minimum, that is when the body is oriented with the maximum principal value of $\underset{\sim}{\alpha}_m$ parallel to $\underset{\sim}{H}_o$. According to equation (59), in the isotropic and linear case this corresponds to the direction of the largest semi-axis both for the diamagnetic and paramagnetic case[58,59].

On the other hand, spherical bodies of anisotropic material will tend to rotate in such a way that they align with the field their largest principal value of susceptibility in the paramagnetic case, and the smallest in the diamagnetic case.

Torque measurements are a standard method for the determination of paramagnetic anisotropy[60]. Equation (63) is the foundation of the standard Faraday's method for the determination of susceptibilities[61].

5. CONDUCTORS AND SUPERCONDUCTORS

In accordance with the ideas to be discussed below this is the case of what may be called perfectly polarisable materials.

For isotropic dielectrics in constant applied fields equations (48) and (46) show that the minimum energy (equilibrium) state is obtained when P, and therefore χ_e, is a maximum. Microscopically, speaking an increase in P corresponds to an increase of the separation between the baricenters of the nuclear and electronic charges. If one imagines a gradual decrease of the restoring forces which confine the electrons around their nuclei, eventually the electrons will be able to experiment macroscopic displacements. Therefore a conductor may be imagined as the limiting case of a linear and isotropic dielectric with $\chi_e = \infty$.

For superconductors the analogy should be drawn not with paramagnets where the polarisation effects are due to the preferential orientation of permanent magnetic dipoles but with diamagnetic materials where they originate from molecular currents induced by the external field. Equations (61) and (59) and the negative sign of the diamagnetic susceptibility χ_m show that in this case the equilibrium state is obtained when χ_m is a minimum.

As the magnetic permeability μ should always be positive[62],

$$\mu = \mu_o (1 + \lambda' \chi_m) > 0 ,$$

therefore

$$\chi_m > -1/\lambda' .$$

These considerations lead one to assimilate -at least from the static point of view[63]- superconductors with perfect diamagnets where $\chi_m = -1/\lambda'$.

The usual definition of a polarisation involves a spatial average of the dipolar moment per unit volume, where the averaging should be done over volumes sufficiently large as to contain many atoms but small enough from the macroscopic point of view[64]. Therefore strictly speaking one may not define polarisations in the conducting and superconducting case. Nevertheless, from what follows it will be seen that when the final expressions are obtained, from a purely mathematical standpoint everything happens as if certain auxiliary vectors there appearing were actual polarisations. Moreover these fictitious polarisations, as well as the rest of the equations, would be obtained if one would consider a conductor as a dielectric with $\chi_e = \infty$, and a superconductor as a diamagnet with $\chi_m = -1/\lambda'$. This provides the mathematical justification of the above drawn analogy. Rigorously, one should not use this analogy as a starting point but should instead prove it[65].

Under the influence of a uniform applied field $\tilde{E}_o$ a surface charge density $\sigma(\tilde{r})$ is induced on a conducting ellipsoid, its contribution to the electrostatic potential

being

$$\phi_{ind}(\underset{\sim}{r}) = k_1 \int_S \frac{\sigma(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\, dS.$$

As in the case of equation (24), one may write

$$\sigma(\underset{\sim}{r}) = \frac{1}{4\pi k_1}\, \underset{\sim}{s}(\underset{\sim}{r}).(\underset{\sim}{E}^{+}(\underset{\sim}{r}) - \underset{\sim}{E}^{-}(\underset{\sim}{r})), \tag{64}$$

where all symbols have the same meaning as in section 3. Therefore, the macroscopic field $\underset{\sim}{E}$ satisfies the integral equation

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o - \nabla\phi_{ind}(\underset{\sim}{r})$$

$$= \underset{\sim}{E}_o - \frac{1}{4\pi}\nabla\int_S \frac{(\underset{\sim}{E}^{+}(\underset{\sim}{r}') - \underset{\sim}{E}^{-}(\underset{\sim}{r}')).d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\,. \tag{65}$$

As for the dielectric case the Ansätz

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o}\, \underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{Q} \tag{66}$$

is taken with $\underset{\sim}{Q}$ a constant vector. Upon substitution of equation (66) in the integrand of equation (65) it is obtained

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_o + \frac{\lambda}{4\pi\varepsilon_o} \nabla \int_S \frac{d\underset{\sim}{S}.(\underset{\sim}{n}^+(\underset{\sim}{r}') - \underset{\sim}{n}^-(\underset{\sim}{r}')).\underset{\sim}{Q}}{|\underset{\sim}{r} - \underset{\sim}{r}'|}$$

$$= \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{Q}, \qquad (67)$$

where in the last step use has been made of equation (33). Therefore the field given by equation (66) is the sought after solution. The auxiliary vector $\underset{\sim}{Q}$ is now determined from the condition that the internal electric field $\underset{\sim}{E}_{int}$ must vanish,

$$\underset{\sim}{E}_{int} = \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{N}.\underset{\sim}{Q} = 0. \qquad (68)$$

For ellipsoids of finite dimensions where $\underset{\sim}{N}$ is non-singular[66],

$$\underset{\sim}{Q} = \frac{\varepsilon_o}{V} \underset{\sim}{\alpha}_c . \underset{\sim}{E}_o , \qquad (69)$$

where

$$\underset{\sim}{\alpha}_c = \frac{V}{\lambda} \underset{\sim}{N}^{-1} \qquad (70)$$

is the conductor's polarisability tensor[10]. It is seen

from equation (46) that

$$\lim_{\chi_e \to \infty} \underset{\sim}{\alpha}_e = \underset{\sim}{\alpha}_c . \tag{71}$$

In order to complete the analogy between conductors and perfectly polarisable dielectrics it will be proved that

$$\underset{\sim}{p} = V\underset{\sim}{Q} \tag{72}$$

is the body's electric dipole moment. From the definition and equations (64), (66) and (25) it follows that[38]

$$\underset{\sim}{p} = \int_S \underset{\sim}{r}\, \sigma(\underset{\sim}{r}) dS = \frac{1}{4\pi k_1} \int_S \underset{\sim}{r}(\underset{\sim}{E}^+(\underset{\sim}{r}) - \underset{\sim}{E}^-(\underset{\sim}{r})).d\underset{\sim}{S}$$

$$= \frac{\lambda}{4\pi k_1 \varepsilon_o} \int_S \underset{\sim}{r}\, d\underset{\sim}{S}.\, (\underset{\sim}{n}^-(\underset{\sim}{r}) - \underset{\sim}{n}^+(\underset{\sim}{r})).\underset{\sim}{Q}$$

$$= \int_S \underset{\sim}{r}\, \underset{\sim}{Q}.d\underset{\sim}{S}. \tag{73}$$

From Gauss-Ostrogradsky's theorem

$$p_j = \int_S x_j\, \underset{\sim}{Q}.d\underset{\sim}{S} = \int_V \nabla.(x_j \underset{\sim}{Q})\, d^3\underset{\sim}{r}$$

$$= \int_V Q_j \, d^3\underset{\sim}{r} \, ,$$

from which equation (72) is inmediately proved. The energy, force and torque equations are the same as in the dielectric case but with the polarisability tensor equation (70). Because from equations (66) and (68) it is found that

$$\underset{\sim}{E}^+ = \underset{\sim}{E}_o - \frac{\lambda}{\varepsilon_o} \underset{\sim}{n}^+ . \underset{\sim}{Q} = \frac{\lambda}{\varepsilon_o} \underset{\sim}{\sim} \underset{\sim}{N} . \underset{\sim}{Q} - \frac{\lambda}{\varepsilon_o} \underset{\sim}{n}^+ . \underset{\sim}{Q}$$

$$= \frac{\lambda}{\varepsilon_o} (\underset{\sim}{n}^- - \underset{\sim}{n}^+) . \underset{\sim}{Q} \, ,$$

it follows from equation (27) that $\underset{\sim}{E}^+$ is normal to the ellipsoid's surface, as it should be.

For the superconducting case, according to the Meissner effect,[63] in applied fields below a certain critical value the internal magnetic induction must vanish,

$$\underset{\sim}{B}_{int} = 0. \qquad (74)$$

This condition is achieved through the induction of conduction currents localized in a very narrow region near the body's surface. If one is interested only in macroscopic effects these currents may be described by a surface current density $\underset{\sim}{j}$. As the magnetisation $\underset{\sim}{M}$, and therefore $\underset{\sim}{H}$, have no physical significance for superconductors[67] the whole discussion should be made in terms of $\underset{\sim}{B}$. The contribution

of the surface currents to the vector potential is[68]

$$\underset{\sim}{A}_m = \frac{k_2}{k_3} \int_S \frac{\underset{\sim}{j}(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \, dS$$

Relating $\underset{\sim}{j}$ to the discontinuities of $\underset{\sim}{B}$ across the body's surface S[68],

$$\underset{\sim}{j}(\underset{\sim}{r}) = \frac{k_3}{4\pi k_2} \underset{\sim}{s}(\underset{\sim}{r}) \times (\underset{\sim}{B}^+(\underset{\sim}{r}) - \underset{\sim}{B}^-(\underset{\sim}{r})) \, , \tag{75}$$

the following integro-differential equation may be written for $\underset{\sim}{B}$,

$$\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}_o + \frac{1}{4\pi} \nabla \times \int_S \frac{d\underset{\sim}{S} \times (\underset{\sim}{B}^+(\underset{\sim}{r}') - \underset{\sim}{B}^-(\underset{\sim}{r}'))}{|\underset{\sim}{r} - \underset{\sim}{r}'|} \, . \tag{76}$$

For all exterior points the following Ansätz is taken

$$\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}_o - \mu_o \lambda' \, \underset{\sim}{n}(\underset{\sim}{r}) \cdot \underset{\sim}{0}, \quad \text{if} \quad \underset{\sim}{r} \notin V. \tag{77}$$

From equations (74) and (31),

$$\underset{\sim}{s} \times (\underset{\sim}{B}^+ - \underset{\sim}{B}^-) = \underset{\sim}{s} \times \underset{\sim}{B}^+ = \underset{\sim}{s} \times (\underset{\sim}{B}_o - \mu_o \lambda' \, \underset{\sim}{n}^+ \cdot \underset{\sim}{0})$$

$$= \underset{\sim}{s} \times (\underset{\sim}{B}_o - \mu_o \lambda' \underset{\sim}{N}.\underset{\sim}{0} + \mu_o \lambda' \underset{\sim}{s} (\underset{\sim}{s}.\underset{\sim}{0}))$$

$$= \underset{\sim}{s} \times \underset{\sim}{G}, \tag{78}$$

where

$$\underset{\sim}{G} = \underset{\sim}{B}_o - \mu_o \lambda' \underset{\sim}{N}.\underset{\sim}{0}. \tag{79}$$

Upon replacement of equation (78) in (76) it is obtained

$$\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}_o - \frac{1}{4\pi} \nabla \times \left(\underset{\sim}{G} \times \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) . \tag{80}$$

From the theorem of the gradient[32]

$$\int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = \int_V \nabla' \left(\frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) d^3\underset{\sim}{r}' = -\nabla I(\underset{\sim}{r}),$$

and therefore

$$\nabla \times \left(\underset{\sim}{G} \times \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) = -\nabla \times (\underset{\sim}{G} \times \nabla I)$$

$$= (\underset{\sim}{G}.\nabla)\nabla I - (\nabla^2 I)\underset{\sim}{G}.$$

Taking into account equation (19) and

$$(\underset{\sim}{G}.\nabla)\nabla I = -4\pi\, \underset{\sim}{n}.\underset{\sim}{G},$$

it follows that

$$\frac{1}{4\pi}\ \nabla \times \left(\underset{\sim}{G} \times \int_S \frac{d\underset{\sim}{S}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \right) = \begin{cases} -\underset{\sim}{n}.\underset{\sim}{G} + \underset{\sim}{G} & \text{if } \underset{\sim}{r} \in V, \\ -\underset{\sim}{n}.\underset{\sim}{G} & \text{if } \underset{\sim}{r} \notin V. \end{cases}$$

Using the previous result in equation (80) (equation which is valid for all $\underset{\sim}{r}$) it is obtained

$$\underset{\sim}{B}_{int} = \underset{\sim}{B}_o + \underset{\sim}{N}.\underset{\sim}{G} - \underset{\sim}{G} = \underset{\sim}{N}.\ (\underset{\sim}{B}_o + \mu_o\lambda'\ (\underset{\sim}{0} - \underset{\sim}{N}.\underset{\sim}{0})) = 0,$$

that is

$$\underset{\sim}{B}_o - \mu_o\lambda'\ \underset{\sim}{N}.\underset{\sim}{0} + \mu_o\lambda'\ \underset{\sim}{0} = 0. \qquad (81)$$

For exterior points equations (80) and (81) give

$$\underset{\sim}{B}(\underset{\sim}{r}) = \underset{\sim}{B}_o - \mu_o\lambda'\ \underset{\sim}{n}(\underset{\sim}{r}).\underset{\sim}{0}\ , \quad \text{if } \underset{\sim}{r} \notin V,$$

in complete accordance with the Ansätz equation (77). From equations (81) and (28) it is easily seen that $\underset{\sim}{B}^+$ is tangential to the body's surface as it should be.

The physical meaning of vector $\underset{\sim}{0}$ will now be analyzed From equations (79), (81), (78) and (75) it follows that

$$\underset{\sim}{j}(\underset{\sim}{r}) = \frac{k_3}{4\pi k_2}\ \underset{\sim}{s} \times \underset{\sim}{B}^{+} = -\frac{\mu_o \lambda' k_3}{4\pi k_2}\ \underset{\sim}{s} \times \underset{\sim}{0},$$

which is the mathematical connection between a uniform magnetisation $\underset{\sim}{0}$ and its equivalent surface current density $\underset{\sim}{j}$ [68]. Therefore the body's magnetic dipole moment $\underset{\sim}{m}$ is given by

$$\underset{\sim}{m} = V\ \underset{\sim}{0}. \tag{82}$$

From equation (81) $\underset{\sim}{0}$ and $\underset{\sim}{m}$ may be explicity written

$$\underset{\sim}{0} = \frac{1}{V}\ \underset{\sim}{\alpha}_s \cdot \underset{\sim}{H}_o\ , \tag{83}$$

$$\underset{\sim}{m} = \underset{\sim}{\alpha}_s \cdot \underset{\sim}{H}_o\ , \tag{84}$$

where[10]

$$\underset{\sim}{\alpha}_s = \frac{V}{\lambda'}\ \underset{\sim}{(N-1)}^{-1}\ , \tag{85}$$

and from equation (59)

$$\lim_{\chi_m \to -1/\lambda'} \underset{\sim}{\alpha}_m = \underset{\sim}{\alpha}_s\ . \tag{86}$$

If ones defines the internal field $\underset{\sim}{H}_{int}$ from the fictitious magnetisation $\underset{\sim}{0}$ in accordance to the usual relation

$$\underset{\sim}{H}_{int} = \frac{1}{\mu_o} \underset{\sim}{B}_{int} - \lambda' \underset{\sim}{0} = -\lambda' \underset{\sim}{0} = \underset{\sim}{H}_o - \lambda' \underset{\sim}{N}.\underset{\sim}{0} \qquad (87)$$

it then follows that both for exterior and interior points

$$\underset{\sim}{H}(r) = \underset{\sim}{H}_o - \lambda' \underset{\sim}{n}(r).\underset{\sim}{0} , \qquad (88)$$

and everything happens as if the superconductor were a perfect diamagnet[69]. One should always remember that equation (87) is only a mathematical fiction with no connection with physical reality[70].

When using equation (87), $\underset{\sim}{j}$ should be considered to be a bound molecular current, that is one should take $\hat{\nabla} \times \underset{\sim}{H} = 0$ everywhere.

6. CAVITIES

Textbooks on Electromagnetism customarily discuss the behaviour of polarisable bodies immersed in an isotropic[71] dielectric or magnetic medium. This requires the study of bodies with interior cavities. In all those discussions it is assumed explicitly or implicitly that the medium is of infinite extent. Of course one knows that all real bodies are finite so that the condition might be taken to be an euphemistic way of saying that the boundaries are distant

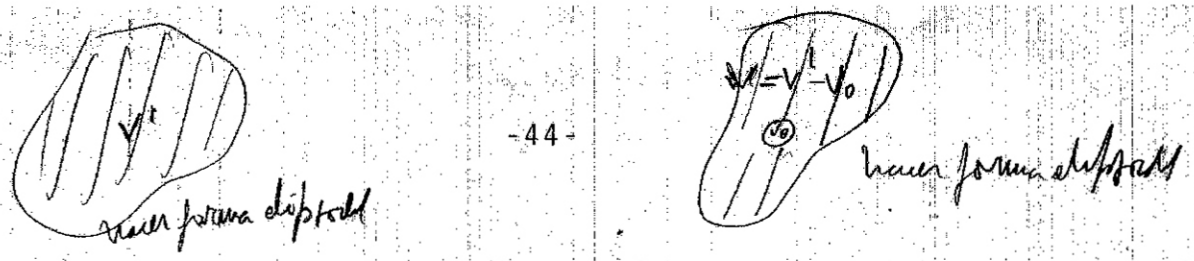
enough so that its presence may be safely ignored. This author has never actually found such a statement in print, but he has often heard it. As a justification the following argument is invoked: 'Dipolar fields fall off with distance as R^{-3}, whereas the number of dipoles at the boundary increases as R^2. Therefore the contributions to the field of the medium's boundaries vanish in the limit of very large R'.

That the argument is fallacious is easily seen from the discussions of the previous sections. For a given material the internal field is a function only of $\underset{\sim}{N}$ and the applied field. As was discussed in section 3, $\underset{\sim}{N}$ is the same for all ellipsoids with semi-axes ca_j where c is variable (similar ellipsoids). In particular one may take the limit $c \to \infty$, and the internal field will still be the same as in the finite dimensions case.

The error in the quoted argument is that the field of a point dipole falls off as R^{-3} but the contribution of a surface element dS falls off as $\Delta\, dS/R^2$, where Δ (see equation (22)) has only angular dependence.

The crucial point is that the internal field *is shape-dependent. Therefore one should not speak of a* medium of infinite extent without specifying its shape. This may be mathematically illustrated for equation (23) where the integral $\underset{\sim}{I}(r)$ equation (2) is now taken to be over all space. In this case $\underset{\sim}{I}(r)$ is found to be conditionally convergent[72] its value depending on the way the limits are taken (shape effects!).

The case will now be considered of an ellipsoidal body -of finite or infinite extent- with an interior ellipsoidal cavity. Let V_o be the volume of the cavity, V' the volume of the body when there is no cavity and $V=V'-V_o$ is what remains of V' after the cavity has been made; the corresponding integrals equation (2) will be called I_o, I' and I. Therefore



$$\underset{\sim}{I}(\underset{\sim}{r}) = \underset{\sim}{I}'(\underset{\sim}{r}) - \underset{\sim}{I}_o(\underset{\sim}{r}),$$

and the depolarisation tensor $\underset{\sim}{n}(\underset{\sim}{r})$ to be asigned to volume V is given in terms of $\underset{\sim}{n}'(\underset{\sim}{r})$ and $\underset{\sim}{n}_o(\underset{\sim}{r})$ by

$$\underset{\sim}{n}(\underset{\sim}{r}) = \underset{\sim}{n}'(\underset{\sim}{r}) - \underset{\sim}{n}_o(\underset{\sim}{r}) \quad . \tag{89}$$

As $\underset{\sim}{n}'$ is constant inside all of V' but $\underset{\sim}{n}_o$ is constant only inside the cavity, $\underset{\sim}{n}$ turns out to be non-uniform inside V. Not much else may be said about the problem in the present context because the application of the formalism requires a constant $\underset{\sim}{N}$ tensor.

It should be mentioned here that a non-uniform tensor $\underset{\sim}{N}(\underset{\sim}{r})$ precludes the existence -at least in homogeneous applied fields- of uniform polarisations and internal fields. That this is so may be seen by considering for instance the magnetic case equation (53). If the interior field $\underset{\sim}{H}_{int}$ were constant, so would be $\underset{\sim}{M}$ in accordance to equation (52). Therefore $\underset{\sim}{X}_m$ could be taken outside the integral sign in equation (53) leading to the solution equation (54). But as $\underset{\sim}{N}(\underset{\sim}{r})$ is not constant this last equation gives a non-uniform interior field $\underset{\sim}{H}_{int}$, which is a contradiction. The same conclusion applies to the dielectric case thus proving the thesis.

A final warning should be issued. When first introduced to the depolarisation tensor method the temptation is felt of solving the case of several ellipsoidal bodies, or of ellipsoidal bodies with cavities, through the use of the superposition principle. That the principle is not applicable follows at once from the non-fixed character of the polarisations.

FOOTNOTES

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[26] Reference 1, p.144.

[27] Reference 14, p.184.

[28] Reference 1, p.48.

[29] See for instance J.Grindlay, An introduction to the phenomenological theory of ferroelectricity, Pergamon Press, 1970, p.119, equation 3.3.89.

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[38] It should be noticed that $4\pi k_1 \varepsilon_o/\lambda=1$ in all standard systems of units (see reference 12).

[39] Reference 14, p.188.

[40] A dyadic operator $\underset{\sim}{ab}$ is such that for any vector $\underset{\sim}{A}$, $\underset{\sim}{ab}.\underset{\sim}{A}=\underset{\sim}{a}(\underset{\sim}{b}.\underset{\sim}{A})$, $\underset{\sim}{A}.\underset{\sim}{ab}=(\underset{\sim}{A}.\underset{\sim}{a})\underset{\sim}{b}$.

[41] Reference 32, p.159.

[42] Reference 34, pp.192-194.

[43] Reference 35, pp.45-60.

[44] M.Abramowitz and I.A.Stegun, editors, Handbook of mathematical functions, Dover Publications, Inc., New York, 1972, p.587.

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[50] Constant means here position-independent as $\underset{\sim}{\chi}_e$ may be, for instance a function of temperature.

[51] Reference 32, p.497-501.

[52] This generates a series whose sum is easily found in the linear case equation (38) as long as one is willing to assume convergence.

[53] Reference 1, p.44, equation (8.10).

[54] Reference 1 , p. 53.

[55] Reference 14, pp. 215-217.

[56] J.Samuel Smart, Effective field theories of magnetism, W.B.Saunders Co., 1966, p.25.

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[61] Reference 4, p.94.

[62] Reference 1, p. 129.

[63] D.Shoenberg, Superconductivity, Cambridge University Press, 1962,pp.14-20.

[64] Reference 12, pp.144, 188.

[65] An opposite procedure is used for superconductors by Landau and Lifshitz,reference 1, p.170.

66 The case where some of the eigenvalues of $\underset{\sim}{N}$ vanish may be treated as the limit of a finite dimensions case, thus showing that the components of $\underset{\sim}{Q}$ vanish in the directions where the semi-axes lengths are infinite. For those cases equation (69) may be easily generalized treating $\underset{\sim}{N}^{-1}$ as a reduced inverse in the subspace of the non-vanishing components of $\underset{\sim}{Q}$.

67 Reference 1, p.168.

68 Reference 14, pp.242-243.

69 This also follows from the condition $\underset{\sim}{B}_{int} = \mu_o(\underset{\sim}{H}_{int} + \lambda' \underset{\sim}{0}) = 0$, which corresponds to $\underset{\sim}{0} = -(1/\lambda')\underset{\sim}{H}_{int}$.

70 Reference 63, pp.20-22.

71 It is known that anisotropic media are not amenable to simple treatments; see for instance the Report of the Coulomb's Law Committee of the A.A.P.T., Am.J.Phys. $\underset{\sim}{18}$, 1 (1950).

72 Reference 35, p.163.

73 Reference 14, pp.38-41.

74 G.Goertzel and N.Tralli, Some mathematical methods of Physics,, McGraw-Hill Book Co., Inc., 1960, pp.52-54.

75 As discussed for the superconducting case; inside the body $\underset{\sim}{H}$ and $\underset{\sim}{D}$ are fictive fields.

76 C.E. Soliverez, IEEE Trans.Magn. $\underset{\sim}{17}$, 1363(1981).