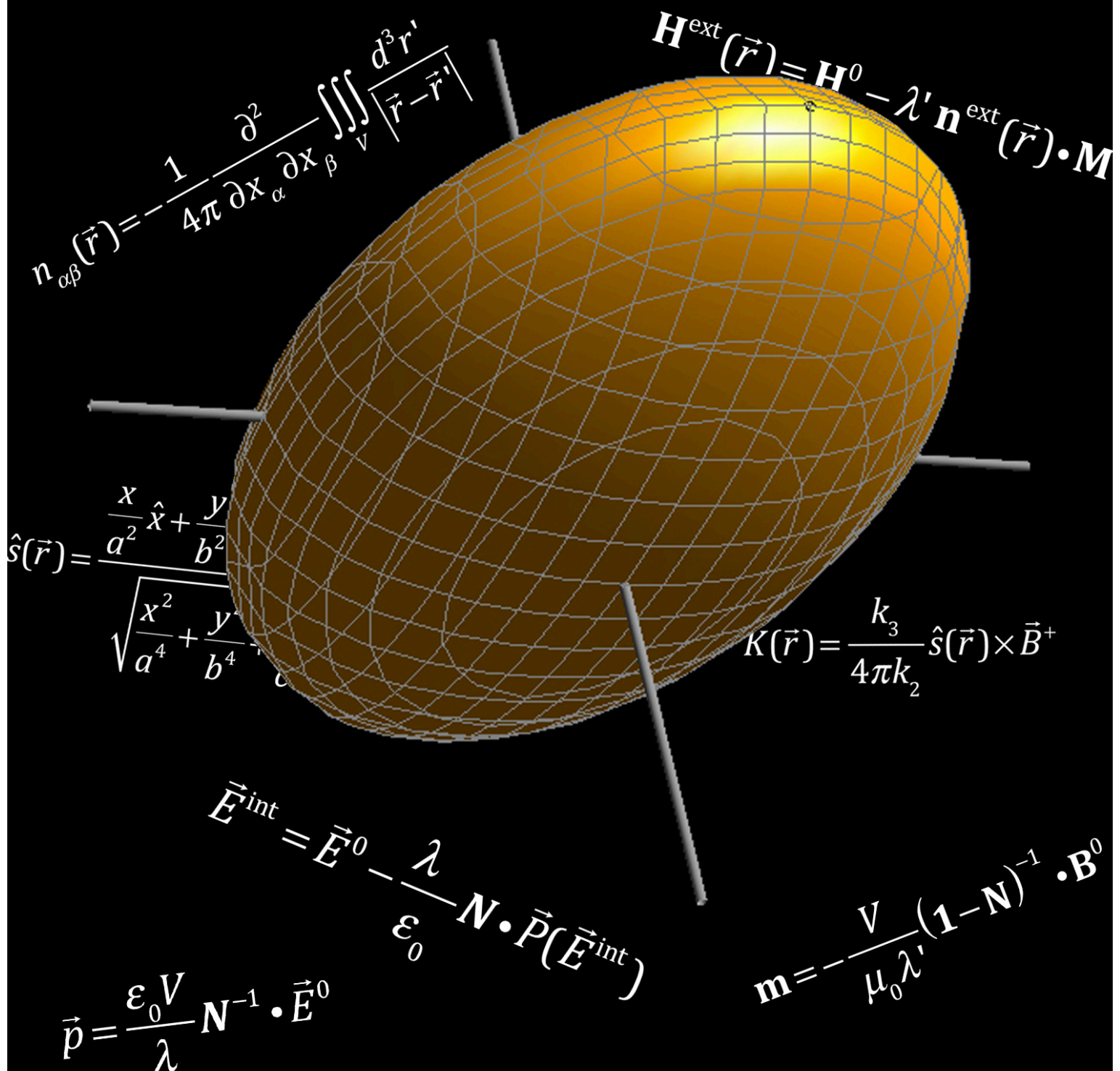


Electrostatics and Magnetostatics of Polarized Ellipsoidal Bodies

The Depolarization Tensor Method

Carlos E. Soliv rez



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**ELECTROSTATICS AND MAGNETOSTATICS
OF POLARIZED ELLIPSOIDAL BODIES:
THE DEPOLARIZATION TENSOR METHOD**

Carlos E. Soliv rez

This work is the English translation by the author of the first edition of his book in Spanish, *Propiedades electrost ticas y magnetost ticas de cuerpos elipsoidales: formalismo del tensor depolarizaci n*. In the process of translation a few errors were corrected and a new chapter, several problems and many appendices were added concerning the actual calculation of the values of the depolarization tensor.

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This work will be modified any time the author finds an error or considers convenient to make further elaborations of its content to make it more useful or easier to undersand. All valuable contributions of readers, specially errors, will be duly acknowledged in the following version of the text.

1st. Spanish edition: Bariloche (R o Negro, Argentina), June 28, 2012, 32.467 words, 147 pages.

1st. English edition: Bariloche (R o Negro, Argentina), July 15, 2016, 39.030 words, 197 pages. ISBN 978-987-28304-0-3.

Abstract

A study is made of the behaviour of ellipsoidal bodies, with no free currents and charges, under uniform applied electrostatic and magnetostatic fields. The method is valid for all sorts of solid homogenous isotropic or anisotropic materials: dielectric, ferroelectric, diamagnetic, paramagnetic, ferromagnetic, conducting, superconducting... Expressions are given, in all cases, for the body's internal and external fields, as well as for the free electromagnetic energy and the torques derived from it. Apart from the body's volume and the electromagnetic properties of the material, these expressions depend only on the depolarization tensor \mathbf{n} determined by the two aspect ratios of the ellipsoid. Explicit expressions are given for \mathbf{n} —both in field points internal and external to the body— in terms of elementary functions except for the triaxial ellipsoid where they are Legendre's elliptic functions. In the non-linear range the electric or magnetic polarization is an implicit function of the applied field and the anisotropy of the material, while the external field is an explicit function of the polarization. In the linear range both the polarization and the internal field are explicit functions of the constant internal value of \mathbf{n} inside the body and the isotropic or anisotropic susceptibility tensor χ , and the external field is an explicit function of the polarization and \mathbf{n} . In the isotropic case the limit values $\chi = \infty$ and $\chi = -1$ (SI units) fully describe the behaviour of conductor and superconductor ellipsoids. A discussion is made of some common errors in the treatment of electromagnetic singularities and of the behaviour of bodies of infinite extension and ellipsoidal cavities.

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Chapter 1: Fundamental concepts

Origin

The topics discussed in this book were developed by the author, with long intervals, since the decade of 1970. The first ones were initially presented in the theory of electromagnetism course for students of physics at the Balseiro Institute (National Atomic Energy Commission – National University of Cuyo, Argentina) in combination with general technics for solving Maxwell's differential equations. In the decade of 1990 a full revision was made in order to adapt the formulation to the level of a first course on electricity and magnetism for engineers with basic knowledge of vector analysis (Bariloche Campus, National University of Comahue, Argentina). Only two small portions of this work^{1,2} were published in international scientific magazines with peer review, because editors considered the full text unsuitable for publication because of its length and pedagogical orientation. A thorough revision was made of all the material thus gathered, revision that will continue as long as readers are willing to make corrections and suggest modifications, and the author is able to analyze and incorporate them.

The depolarization tensor method provides an efficient and compact way of solving the case of ellipsoidal bodies polarized by constant electric and magnetic fields. It is thus well suited for technical applications, but also for research on anisotropic properties of materials, for which no general method is, to the knowledge of the author, presently available. From the point of view of the foundations of electromagnetism, it is one of the simplest and clearest example of the inextricable relationship between geometric and physical properties of materials. This relationship is expressed by the boundary conditions that determine the solution of the differential equations, but this is never so clearly and explicitly expressed as by the depolarization tensor. It is also a good illustration of the power of integral methods for solving electromagnetic problems. The main contribution of the author in this field, as shown by the current literature on the subject, is the generalization of the use of this tensor for all static electromagnetic problems, the treatment of anisotropic materials and the calculation of fields outside the body.

History

The use of what is today identified as *depolarization tensor* is very old in the analysis of permanently magnetized materials. The first to derive it explicitly from a potential was probably Maxwell³. To that end he used Poisson's proof (see eq.

¹ Soliv rez (1981), pp. 1363-1364.

² Soliv rez (2008), pp. 203-207.

³ Maxwell, pp. 66-69.

2.2) that the calculation of the field created by a uniformly magnetized body is mathematically equivalent to taking the directional derivative of the gravitational attraction of an homogenous mass distribution with the same shape. The mathematical analysis of the subject thus benefits from the numerous studies of gravitational potential (proportional to the integral f defined by eq. 3.8) made since Newton's times. It is in this field of physics that ellipsoidal bodies were and still are intensively studied⁴.

Maxwell proved that ellipsoidal bodies are uniformly polarized when immersed in a uniform static applied field (a constant vector in the region occupied by the body). He points out for integral f (there called V^5)

... the only cases with which we are acquainted in which V is a quadratic function of the coordinates within the body are those in which the body is bounded by a complete surface of the second degree, and the only cases in which such a body is of finite dimensions is when it is an ellipsoid.

The origin of this behaviour —which is analyzed in Chapter 2— is that internal fields, and the polarizations they produce, are proportional to second derivatives of f . Soon afterwards Thomson and Tait proved ellipsoids to be the only finite bodies with this property⁶.

The principal values of Maxwell's entity are frequently called demagnetizing or demagnetization factors or coefficients, and the study of its history began as early as 1896⁷. The author of this book has not been able to determine who, when and where rebaptized it as *depolarization tensor*, a name that rightly includes both electric and magnetic phenomena⁸. In 1941, Stratton's classical text on electromagnetism introduced the principal values of the depolarization tensor inside dielectrics by solving Laplace's equation in ellipsoidal coordinates. He called them *depolarization factors*⁹ but he neither mentions their tensorial character (which Maxwell apparently did not detect), nor applies them to conducting or magnetic materials. Two distinguished russian theoretical physicists, L. Landau and E. Lifschitz, generalized in 1969 the use of these tensorial coefficients to the case of dielectrics, conductors and superconductors in volume 8 of their monumental Course on Theoretical Physics¹⁰, introducing them in a similar fashion

⁴ N. R. Libovitz, *The mathematical development of the classical ellipsoids*, International Journal of Engineering Science vol. **36**, pp. 1402-1420 (1998).

⁵ Maxwell, p. 67.

⁶ W. Thomson, W & P. G. Tait, P. G., *Treatise on Natural Philosophy*, vol. 2, Cambridge University Press Cambridge (England), 2009 (new printing of the 1883 edition). P. Dive, *Attraction des ellipso  des homog  nes et r  ciproque d'un th  or  me de Newton*, Bull. Soc. Math. France vol. **59**, pp. 128-140 (1931). W. Nikliborc, *Eine Bemerkung   ber die Volumpotentiale*, Math. Z. vol. **35**, 625-631 (1932).

⁷ H. Du Bois, *The Magnetic Circuit*, Longmans (1896).

⁸ The name was used by Van Vleck, chapter IV, in 1932.

⁹ Stratton, pp. 206 and 213.

¹⁰ Landau and Lifchitz: conductors (pp. 7, 25, 28), dielectrics (pp. 42-45, 54), ferromagnets (p. 163) and superconductors (p. 170).

as Stratton¹¹. They seem to be the first ones to point out the mathematical equivalence of the behaviour of conductors and dielectrics with infinite susceptibility¹². In 1945 Stoner gave graphs and tables of the demagnetizing factors for spheroids and general ellipsoids, without mentioning the tensor's general properties. Almost simultaneously, Osborn gave a number of useful formulas, tables and graphs for obtaining the values of the demagnetizing coefficients. In 1966 Moskowitz and Della Torre analyzed some general properties of the depolarization tensor¹³. In 1981 this author proved, for the magnetic case, that this tensor also solves the anisotropic case and provides the non-uniform value of the induced field outside the body¹⁴. In 2006 alternative methods were proposed for the calculation of the tensor¹⁵. In 2008 this author discussed the application to the electric case, including electrets and conductors¹⁶.

Applications

Elementary texts on electricity and magnetism solve only the behaviour of the following fictive bodies (the equivalent ellipsoids are identified between parenthesis):

- point charge (ellipsoid with 3 equal and very small semi-axes);
- sheet of uniform thickness and infinite extension (ellipsoid with 1 finite semi-axis and 2 very large semi-axes);
- cylinder of circular cross-section and infinite length (ellipsoid with 2 equal semi-axes and 1 very large semi-axis);
- volume of infinite extension (ill defined problem discussed in [Chapter 5](#)).

The sphere —the finite body of highest symmetry— is solved only in advanced courses of electromagnetism by reducing Laplace's equation to a set of ordinary differential equations with the canonical method of separation of variables in spherical coordinates¹⁷. Solving triaxial ellipsoids (three different semi-axes) with the same method requires advanced knowledge of metric properties and the use of an uncommon and complex system of curvilinear coordinates¹⁸.

On the other hand, the depolarization tensor method requires only a basic knowledge of vector analysis in order to solve the whole family of ellipsoids, including the sphere, sheets of infinite extension, circular and elliptic cylinders of infinite length. The internal fields may thus be expressed in terms of constant matrices peculiar of each body, whose geometric part is a function of two parameters which are the ratios of two semiaxes to the third (aspect ratios).

¹¹ Landau and Lifchitz, p. 20.

¹² Landau and Lifchitz, p. 40.

¹³ Moskowitz and Della Torre, pp. 739-744.

¹⁴ Soliv rez (1981).

¹⁵ M. Beleggia, M. De Graef and Y. Millev, *Demagnetization factors of the general ellipsoid: An alternative to the Maxwell approach*, Philosophical Magazine vol. **86**, pp. 2451-2466, 2006.

¹⁶ Soliv rez (2008).

¹⁷ Jackson, pp. 156-160; Reitz, pp. 93-94; Stratton, pp. 205-207.

¹⁸ Landau and Lifchitz, pp. 20-27; Stratton pp. 58-59, 211-213, 257-258.

Only limited to the homogeneous case, the method is valid for any of the following types of materials:

- isotropic ones, like amorphous and polycrystalline solids;
- anisotropic ones, like single crystals;
- permanent and induced electric (electrets, ferroelectrics...) and magnetic (ferromagnets, antiferromagnets...) polarizations;
- conductors (metals) and insulators (dielectrics);
- superconductors.

The method allows to easily solve a large range of practical problems for realistic finite bodies by using mostly elementary mathematical function and a basic knowledge of vector analysis. The exception are triaxial ellipsoids which — due to the appearance of elliptic functions— require the use of tables or software like Mathematica® or Maple®.

Experimental determination of polarizations¹⁹ and of electric and magnetic anisotropy constants are usually made by measuring forces and torques on spheres, disks and cylinders placed in known uniform applied fields or appropriate non-uniform ones²⁰. Inversely, these forces and torques provide the value of the field in which the bodies are immersed. Ellipsoids include or are good approximations to those bodies²¹, also allowing an estimation of the errors caused by departures from ideal shapes. They are also used in single crystals for the calculation of lattice sums²², the microscopic contributions of atomic and molecular electric charges and dipole moments to the macroscopic field. There are numerous technological methods and properties —like magnetic resonance²³, electrostatic precipitation, magnetic coercivity of tapes and disks²⁴— that can be more easily interpreted when using ellipsoidal bodies. Also, some experiments suggest that in certain cases ellipsoidal particles may appear spontaneously²⁵. Widely used anisotropic materials, seldom discussed in technical textbooks²⁶ because of the difficulties of its mathematical treatment²⁷, may be easily discussed with the depolarization tensor method.

It is, therefore, surprising that the electromagnetic behaviour of ellipsoidal bodies is absent from most textbooks on electricity and magnetism, being confined

¹⁹ In line with the use of the denomination *depolarization tensor*, a *magnetization* will be called *magnetic polarization* in this book. Therefore, the term *polarization* refers both to the electric and magnetic one.

²⁰ H. Zijlstra, *Experimental Methods in magnetism*, vol. 2, North Holland Publishing Co., Amsterdam (1967).

²¹ M. Beleggia et al., *J. Phys. D: Appl. Phys.* **39**, pp. 891-899 (2006).

²² M. Widom, *Shape-Adapted Ewald Summation*.

²³ J. H. Duyn and T. H. Barbara, *Magn. Reson. Med.*, **72**(1), pp. 1-3 (2014).

²⁴ Q. F. Brown Jr. and A. H. Morrish, *Phys. Rev.* vol. **105**, p. 1198 (1957).

²⁵ G. R. Davies, *The mechanisms of piezoelectricity and pyroelectricity in poly(vinylidene fluoride)*, The Dielectric Society 1984 Meeting, Abstracts of Invited Papers.

²⁶ A notable exception is the book by Landau and Lifchitz.

²⁷ See, for instance, the *Report of the Coulomb's Law Committee of the American Association of Physics Teachers*, *Am. J. Phys.* vol. **18**, p. 1 (1950).

to theoretical treatises and articles in specialized journals. This book intends to fill this gap, collecting scattered formulas and properties, generalizing the formulas and adding new ones whenever possible.

Requirements

The reader is required to be familiar with the differential and integral physical laws relating electric and magnetic fields and potentials with their sources: electric charge density and polarization, current density and magnetization. The required knowledge of mathematics includes operations with matrices, derivatives, surface and volume integrals, the combined application of vectorial operators and their integral theorems. As previously mentioned, it is not necessary to solve Laplace's equation, an unavoidable requisite unless the depolarization tensor method is used.

Limitations of the method

The depolarization tensor method is valid only when:

- The applied field is uniform over the whole body, but not necessarily elsewhere. In the magnetic case the condition can be easily obtained with Helmholtz coils²⁸.
- The body's material must have a stable ellipsoidal shape—which rules out liquids)—and uniform electric or magnetic properties in a macroscopic scale. Otherwise, the material may be isotropic (amorphous or micro-polycrystalline) or anisotropic (single crystal or textured polycrystalline²⁹).

The applied field does not need to be fixed (unmodifiable by the introduction of the body), except for simplicity in the calculation of the energy and the resulting forces and torques. The method is valid for torque experiments whenever the value of the applied field can be measured or calculated after the introduction of the ellipsoidal body.

Dealing with singularities

Electromagnetic phenomena are fully described by Maxwell's equations and the auxiliary constitutive equations which describe material properties. The standard formulation makes use of vector analysis and techniques of resolution of partial differential equations, but not of integro-differential equations like those given by eqs. 2.31. The depolarization tensor formulation dispenses with the use of differential equations, but makes intensive use of vector analysis and of integral relationships like the divergence theorem, where special precautions must be taken when dealing with singularities like point charges and jump discontinuities across the body's surface:

²⁸ W. Franzen, *Generation of Uniform Magnetic Fields by Means of Air-Core Coils*, Rev. Sci. Instr. vol. **33**, pp. 933-938 (1962).

²⁹ A polycrystalline material is textured when its microcrystals have predominance of certain type of faces. This often happens in laminated metallic materials, because the lamination process favours the appearance of certain crystal planes and disfavours others.

point charge type singularity: $1/|\vec{r}-\vec{r}'|$ cuando $\vec{r} \rightarrow \vec{r}'$, (1.1)

jump discontinuity: $\vec{F}(\vec{r}^+) \neq \vec{F}(\vec{r}^-)$ through surface S . (1.2)

These singularities appear repeatedly in surface and volume integrals, both for the electric and magnetic case. As they arise from a simplified mathematical representation of physical facts, to solve them is necessary to trace them back to the basic physical phenomena they describe. This physical analysis is made in the two following sections, while in [Appendix 3](#) the integral theorems of vector analysis are carefully reformulated in order to deal correctly with these singularities, where it may not be valid, for instance, to interchange the order of derivation and integration.

Scalar and vectorial magnitudes with constant values within a bounded region play an essential role in the method. Electric and magnetic polarization, for instance, have a finite and constant value inside and vanish outside the ellipsoidal volume V . Applied electric and magnetic fields are constant only in region that barely exceeds V , and may vary widely in the outside. The jump discontinuities of polarization through the body's surface are the sources of the internal and external induced field. The invariability of applied fields in V play a crucial role in the determination of the forces and torques exerted on the ellipsoidal body (see [Chapter 5](#))

It is therefore convenient to have a common denomination for these kind of fields, in order to clearly differentiate them from the constant vectors of mathematics that are invariable over all space. For that reason, in this book a vector is said to be uniform in a region V when it is constant in V (invariable magnitude, direction and sense), but not necessarily outside.

Point charge type singularity

The origin of the point charge type singularity is the experimental Coulomb's law (the value of constant k_1 in different systems of units is discussed in [Appendix 1](#)),

$$\vec{F}(\vec{r}) = k_1 q q' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = q \vec{E}(\vec{r}), \quad (1.3)$$

which gives the force experienced by a charge q in presence of a charge q' . The experiment requires two different charges: the field's source q' and the charge q where the force is applied (unless one is willing to invoke a "M  nchausen effect"³⁰). It is therefore necessary to differentiate between the *field point* \vec{r} and the *source point* \vec{r}' . This requirement is ignored in the macroscopic formulation of electromagnetism because otherwise fields would not be defined inside continuous charge distributions. The price one has to pay for this simplification is that the resulting macroscopic field differs from the microscopic field actually experienced by atoms and molecules.

³⁰ The mythical M  nchausen baron raised himself by pulling from his boot's straps.

Coulomb's law gives origin to the definition of the differential electric field $d\vec{E}(\vec{r})$ created at \vec{r} by an infinitesimal charge $dq = \rho(\vec{r}')d^3\vec{r}'$ at \vec{r}' . This field is expressed in terms of the volume charge density ρ by

$$d\vec{E}(\vec{r}) = k_1 \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} dq = k_1 \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d^3\vec{r}', \quad (1.4)$$

where field and source points are explicit, as will be always done in all integrals herein.

The total electric field created by the complete charge distribution ρ is thus given by

$$\vec{E}(\vec{r}) = k_1 \iiint_V \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d^3\vec{r}'. \quad (1.5)$$

Expressions 1.3 and 1.5 do look similar, but they differ in a fundamental way. While for Coulomb's law it is not valid to take $\vec{r} = \vec{r}'$, the electric field expression requires its validity for all volume and surface integrals. There are additional problems because the volume charge density ρ for point charges is not a well behaved function, and the same happens with surface charge densities. These singular densities may be given mathematical rigour by the use of distribution theory, a method that will not be used here for the evaluation of integrals of this kind because most engineers are not familiar with it. Instead, appropriate care will be taken for the evaluation of singular integrands with standard methods, an example of which is the calculation of the trace of the depolarization tensor in eqs. 3.15 and A3.7.

The simplest case of dealing with point charge type singularities is that of eq. 1.5. One may take a system of spherical coordinates with origin at point \vec{r} so that the integral becomes

$$\vec{E}(\vec{r}) = -k_1 \int_0^{2\pi} d\varphi \int \sin\theta d\theta \int_0^{\hat{R}(\varphi,\theta)} \frac{\hat{R}}{R^2} \rho(\varphi,\theta,R) R^2 dR \quad \text{con} \quad \vec{R} = \vec{r}' - \vec{r}. \quad (1.6)$$

The singularity turns out to be integrable because of the factor R^2 in the differential volume element. As will be discussed later on, this does not mean that such singularities do not affect the validity of the standard divergence and curl theorems of vector calculus.

Let's examine the first of these theorems. Coulomb's Law eq. 1.3 is the origin of Gauss's Law

$$\oiint_S \vec{E}(\vec{r}) \cdot d\vec{S} = 4\pi k_1 Q_S, \quad (1.7)$$

where Q_S is the charge inside the closed surface S . According to the divergence theorem eq. A3.8,

$$\oiint_S \vec{E}(\vec{r}) \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{E}(\vec{r}) d^3\vec{r} = 4\pi k_1 Q_S = 4\pi k_1 \iiint_V \rho dV, \quad (1.8)$$

where V is the region bounded by S . The result is one of Maxwell's equations (see eq. A1.3)

$$\nabla \cdot \vec{D}(\vec{r}) = \epsilon_0 \nabla \cdot \vec{E}(\vec{r}) = 4\pi k_1 \epsilon_0 \rho, \text{ so that } \nabla \cdot \vec{E}(\vec{r}) = 4\pi k_1 \rho. \quad (1.9)$$

The volume charge density ρ of a point charge eq. 1.3 would then be

$$\begin{aligned} \rho &= \frac{1}{4\pi k_1} \nabla \cdot \vec{E}(\vec{r}) = \frac{q'}{4\pi} \nabla \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) \\ &= \frac{q'}{4\pi} \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \nabla \cdot (\vec{r} - \vec{r}') + \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \right) \cdot (\vec{r} - \vec{r}') \right) \\ &= \frac{q'}{4\pi} \left(\frac{3}{|\vec{r} - \vec{r}'|^3} - \frac{3(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^5} \right) = 0, \end{aligned} \quad (1.10)$$

where use has been made of eq. A2.5.

The experimental validity of Gauss's Law has been thoroughly verified³¹, so it follows that eq. 1.9 does not hold. Some authors try to "solve" the problem by denying the validity of eq. 1.10 with dubious mathematical arguments³², but the origin of the paradox is that the divergence theorem eq. A3.8 is not valid for singular fields like the one given by eq. 1.3.

A similar problem arises with the calculation of the following expression, that will be used later on for the calculation of the depolarization tensor trace:

$$\nabla \cdot \nabla \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|}. \quad (1.11)$$

This expression is well known in physics, where it is usually solved by using Dirac's delta function, that is, invoking distribution theory. Some authors of texts in electromagnetism justify its value by juggling with the order of derivation and integration, frequently making invalid manipulations. The topic is discussed in Appendix 3 in order to make a valid derivation of eq. A3.7 with no recourse to Dirac's delta function.

One may deal with singular fields in two different ways. The first one, which preserves Maxwell's equations even for singular charge distributions, is to extend the definition of charge (and current) densities in order to include point charges, surface and line charge distributions (and the corresponding current distributions). This requires distribution theory, a complex mathematical

³¹ See, for instance, Young and Freedman, p. 721.

³² Reitz, p. 45, ignores the basic rule of making all possible simplifications before taking limits, not afterwards.

formalism³³ well known to physicists but seldom used by engineers. The other way—discussed in [Appendix 3](#)—is to extend the validity of the integral theorems of vector analysis for the case of singular charge distributions like point charges. The latter approach is used by some texts on electromagnetism, but usually not explicitly stated as a generalization of the standard theorems.

Step discontinuities

Step discontinuities present similar difficulties. In a well known text on electromagnetism which the author used for his first studies of the subject, the following integration by parts is made³⁴ (here rewritten in our notation):

$$\begin{aligned} & \iiint_{V'} \nabla \Phi(\vec{r}) \cdot (\vec{D}(\vec{r}) - \vec{D}^0(\vec{r})) d^3r \\ &= \iiint_{V'} \nabla \cdot (\Phi(\vec{D} - \vec{D}^0)) d^3r - \iiint_{V'} \Phi(\vec{r}) \nabla \cdot (\vec{D}(\vec{r}) - \vec{D}^0(\vec{r})) d^3r, \end{aligned} \quad (1.12)$$

where Φ is an electric potential, \vec{D} , \vec{D}^0 are the displacement vectors in the presence and absence of the material body, and use has been made of eq. [A2.5](#). The integration volume V' includes both the dielectric body and the sources of Φ . From divergence theorem eq. [A3.8](#), Jackson argues that, as the integration surface S' is outside the body, it follows that there $\vec{D} = \vec{D}^0$ and

$$\iiint_{V'} \nabla \cdot (\Phi(\vec{D} - \vec{D}^0)) d^3r = \oint_{S'} \Phi(\vec{D} - \vec{D}^0) d^2\vec{r} = 0, \quad (1.13)$$

where $d^2\vec{r}$ is the differential element of area vector normal to the surface.

The standard divergence theorem may be used when the the first partial derivatives of the integrand are continuous in V' ³⁵, but $\vec{D} - \vec{D}^0$ has a step discontinuity across the body's surface S . One may avoid the problem by dividing the volume of integration in two by the surface S of step discontinuities, the method used in the formulation of the generalized divergence theorem eq. [A3.18](#). Using this theorem eq. [\(1.12\)](#) gives

$$\iiint_{V'} \nabla \cdot (\Phi(\vec{D} - \vec{D}^0)) d^3r = \oint_{S'} \Phi(\vec{D} - \vec{D}^0) d^2\vec{r} - \oint_S \Phi(\vec{D}^+ - \vec{D}^-) d^2\vec{r} \neq 0, \quad (1.14)$$

because the last integrand does not vanish.

This sort of problems, which appear frequently in texts on electromagnetism, usually remain undetected because such complex integrals are seldom computed for real distributions of matter. This was not our case because it was quite easy to

³³ For a simplified formulation based on L. Schwartz, *Theorie des distributions* (Hermann, Paris, 1950) see R. J. Gagnon, Am. J. Phys. vol. **38**, pp. 879-891 (1979).

³⁴ Jackson, p. 125.

³⁵ Korn and Korn, p. 163.

use the depolarization tensor method to compare the initial expression for an ellipsoidal body with the final one, finding no match.

Physical units and mathematical notation

Texts used by physicists —like Jackson's— tend to use the Gauss system of units because it's more suited for relativistic analysis, while engineers uniformly use standard SI units. Instead of committing ourselves to one of the two, we use the general system introduced by Jackson through constants k_1, k_2, k_3, λ and λ' defined and tabulated in [Appendix 1](#) for all commonly used systems of units³⁶.

The mathematical notation used in this book uses the following conventions:

- Vectors (fields, polarizations...) and tensors (depolarization, susceptibilities...) are represented in two alternative ways:
- 1. As linear combinations of products of components and unit vectors such as $\vec{E}(\vec{r}) = E_z(\vec{r})\hat{z}$. Variables are always written with italics, vectors are always crowned with arrows and unit vectors with $\hat{\cdot}$. The omission of the arrow indicates the vector module, $|\vec{E}| = E$. Second rank tensors are represented with italics and bold letters, and its components are given in dyadic notation³⁷ (see [Appendix 6](#))

$$\mathbf{n}(\vec{r}) = \sum_{\alpha} \sum_{\beta} n_{\alpha\beta}(\vec{r}) \hat{x}_{\alpha} \hat{x}_{\beta}, \quad (1.15)$$

thus making clear that its scalar product with a vector gives another vector. This intuitive and compact notation simplifies the derivation of equations, but not its resolution. For the latter it is more convenient to use the following alternative notation.

- 2. Vectors are alternatively represented as column or row matrices of dimension 1×3 or 3×1 . Second rank tensors are then represented as square 3×3 matrices, as in eq. 3.66. The only disadvantage of this representation is that depends on the chosen coordinate system, but so does the diagonal form (see below) of the depolarization tensor.
- V denotes both the region occupied by the ellipsoidal body (in integrals) and the magnitude of its volume (in equations), concepts easily discriminated from the context.
- S is the boundary surface of region V (the body) and $\hat{s}(\vec{r}^S)$ the outgoing unit vector normal to S at point \vec{r}^S .
- x, y, z are orthogonal cartesian coordinates.
- Different coordinates are generically called x_{α} , where $\alpha = x, y, z$ and $x_x = x, x_y = y, x_z = z$.

³⁶ Jackson, pp. 613-618.

³⁷ Morse and Feshbach, pp. 54-92.

- Unless explicitly stated otherwise, the cartesian coordinate system in use is the one that diagonalizes the matrix representation of the depolarization tensor, the ellipsoidal's body principal system of eq. A7.2.

- $\delta_{\alpha\beta}$ is Kronecker's delta function (or matrix) defined by $\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$.

Therefore, the unit vectors corresponding to each of the orthogonal coordinate axis fulfill the condition $\hat{x}_\alpha \cdot \hat{x}_\beta = \delta_{\alpha\beta}$.

- Symbol $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ is the gradient operator nabla such that its application to a scalar function gives a vector. The divergence operator is written as $\nabla \cdot$, the curl as $\nabla \times$ and the laplacian as $\nabla \cdot \nabla$ or Δ .
- \vec{r}' is a source point for the field (electric charges, polarization, electric current...), which is always explicitly distinguished from the field point where the generated field acts upon.
- d^2r' is the scalar differential element of area in a surface integral.
- $d^2\vec{r}'$ is the vectorial differential element of area in a surface integral.
- d^3r' is the scalar differential element of volume in a volume integral.

Organization of the book

Three categories of materials have very similar behaviour both in the electric and magnetic case: spontaneously polarized ones (like electrets, ferroelectrics and ferromagnets); those polarized only in the presence of fields (like dielectrics, diamagnets and paramagnets); those with vanishing internal fields (conductors and superconductors). In order to facilitate comparisons, in [Chapter 2](#) these cases are separately discussed for electric and magnetic substances.

In permanently polarized materials a uniform applied field is necessary only for maintaining and orienting the uniform polarization. The body's internal and external fields are then fully characterized by the depolarization tensor \mathbf{n} , both with or without applied fields.

For substances that get polarized only in the presence of applied fields, the linear approximation is valid except in unusually high fields. In this approximation the induced polarization is proportional to the uniform applied field through a susceptibility that is a scalar in the isotropic case and a second rank tensor χ in the anisotropic one. It is then found that the induced polarization and the internal field — both uniform for ellipsoidal bodies — may be fully expressed in terms of \mathbf{n} , the susceptibility and the applied field. This result is well known for isotropic materials, but has never been fully discussed for the anisotropic case³⁸. In the linear range a tensorial relationship —the body's polarizability, function of \mathbf{n} and χ ³⁹—may be established between the body's electric or magnetic moment and the applied field, just as in the molecular case⁴⁰.

³⁸ See the discussion made by Landau and Lifchitz in pp. 58-61.

³⁹ Landau and Lifchitz, p. 7, introduce the tensor but do not relate it to \mathbf{n} .

⁴⁰ Kittel, p. 459.

According to the standard definition of macroscopic fields, there is no polarization inside a conductor or a superconductor. However, a fictive polarization (in this book called *equivalent polarization*) may be defined such that the problem turns out to be equivalent to the case of induced polarization. The equivalent polarizations gives both the right value of the internal fields and the body's electric and magnetic moments. For conductors the corresponds to the limit of a perfect dielectric, that of infinite electric susceptibility. For a superconductor, it corresponds to the limit of perfect diamagnetism where the diamagnetic susceptibility equals -1 .

The integro-differential equations that determine induced fields and internal polarizations are derived for all cases and collected in [Table 1](#). An ansatz corresponding to the uniform solution satisfies all these equations. At the end of [Chapter 2](#) an iterative method of solving the integro-differential equations is given, which confirms the value of the uniform solutions.

All along [Chapter 2](#) the depolarization tensor and some of its properties are used without proof in order to show its convenience and way of use. Its definition and elementary analysis starts at [Chapter 3](#), where some expressions are given for the tensor, as well as a simple method for determining its principal values in the case of the sphere, the infinitely long circular cylinder and the infinite sheet of constant thickness. [Chapter 4](#) tackles the more complex case of the spheroids, uniaxial ellipsoid and elliptic cylinder, as well as the study of the value of the depolarization tensor outside the body for triaxial ellipsoids.

[Chapter 5](#) gives a (too) brief introduction to the calculation of electromagnetic energies, from which are derived the expressions for forces and torques exerted on ellipsoidal bodies immersed in uniform applied fields. Infinite bodies and cavities are discussed at the end of this chapter, showing the inconsistencies in their standard analysis wich usually invokes a “rigid” behaviour of the polarization. Both topics present difficulties rarely discussed in standard textbooks; they deserve a more thorough analysis that is barely sketched here, probably a whole book.

The inclusion of [Chapter 6](#), which gives detailed solutions to illustrative exercises on all previously discussed topics, requires justification. This book is neither a treatise on electromagnetism, nor one on the theory of polarized bodies. It is an introduction to a practical method of calculation of static electromagnetic properties of ellipsoidal bodies with no recourse to the resolution of Laplace's equation, that is, to the techniques of resolution of partial differential equations. Exercises have, therefore, a central role, but here they are not dispersed through the text as is customary. Two main arguments justify grouping them in a single chapter.

The first one is that the specificity of concrete problems distracts the reader from the main line of reasoning. In the few ocasions where the author felt that the problem is relevant to this reasoning, an actual case is discussed in the pertinent place, but a link is also given in [Chapter 6](#). The necessity of this link is a consequence of the second justification for the existence of this chapter.

This book is conceived as a complement to ordinary texts on electromagnetism because of their customary lack of treatment of finite bodies, usually reserved to courses with advanced knowledge of partial differential equations. Therefore — although the author does not recommend it— it is expected that its use will be

occasional and fragmentary, which requires special provisions. The given organization makes the selection of exercises easier. So does the redundancy in the discussion of central concepts and the provision of a detailed alphabetical index.

Chapter 2: Ellipsoids in Electric and Magnetic Fields

This chapter discusses the uniform polarizations —permanent, induced and equivalent— and the fields generated when homogeneous ellipsoidal bodies are immersed in uniform applied fields. Results are expressed in terms of the depolarization tensor \mathbf{n} , leaving the solution of specific cases for [Chapter 6](#).

Electric polarization

Basic equations

The discussion of the subject first made by the author in 2008⁴¹ is here modified in order to make its understanding easier. The electric polarization of an ellipsoidal body V is usually a function of the position and generates an electric field $\vec{E}_p(\vec{r})$ derivable from a potential⁴²:

$$\phi_p(\vec{r}) = k_1 \iiint_V \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3r', \quad \vec{E}_p(\vec{r}) = -\nabla \phi_p(\vec{r}). \quad (2.1)$$

The formula is the extension of the potential of a point dipole to the case of a continuous distribution, the macroscopic representation of a collection of neutral atoms where the baricenters of the nuclear and electronic charges do not coincide. Poisson was, apparently, the first to notice that

$$\phi_p(\vec{r}) = -(\vec{P} \cdot \nabla) k_1 \iiint_V \frac{1}{|\vec{r} - \vec{r}'|} d^3r', \quad (2.2)$$

The characterization of a uniformly polarized ellipsoidal body may thus be derived from that of a uniformly charged one, a property that will be widely used in this book.

As discussed in p. 7, the integrand in eq. 2.1 has an integrable singularity. It is frequently said that the macroscopic electromagnetic fields are the average of the microscopic fields, but it has been convincingly argued that one should take,

⁴¹ Solivérez (2008).

⁴² Reitz, p. 78 eq. 4-7.

instead, the average of the potential⁴³. The subject, not discussed in this book, is mentioned only because the depolarization tensor may have an important role for determining the connection between those two scales⁴⁴. At the same time, a good understanding of the origin of fundamental expressions as eq. 2.1 (and similar magnetic ones) is the best guide for the interpretation and resolution of mathematical singularities originated in mathematical idealizations of actual non-singular distributions of charge.

The potential eq. 2.1 may be rewritten in two different ways. The first one will be used for the discussion of an ellipsoidal body with permanent or induced electric polarization. The second one, for conductors.

Using eqs. A2.2 and A2.5 the integrand of eq. 2.1 may be rewritten in the following fashion :

$$\begin{aligned}\phi_p(\vec{r}) &= k_1 \iiint_V \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3r' = -k_1 \iiint_V \vec{P}(\vec{r}') \cdot \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r', \\ &-k_1 \iiint_V \left(\nabla \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \right) d^3r' = -k_1 \nabla \cdot \iiint_V \frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'.\end{aligned}\tag{2.3}$$

The last step requires a permutation of the order of derivation and integration, an operation that may not be valid for singular integrands. The problem has been studied for this kind of integrals in the gravitational case and proven to be mathematically valid⁴⁵.

The total electric field $\vec{E}(\vec{r})$ is the vectorial sum of the uniform applied field \vec{E}^0 and the induced field \vec{E}_p :

$$\boxed{\vec{E}(\vec{r}) = \vec{E}^0 - \nabla \phi_p(\vec{r}) = \vec{E}^0 + k_1 \nabla \left(\nabla \cdot \iiint_V \frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' \right)},\tag{2.4}$$

equation valid in all space for all sorts of polarizations. This equation will be the starting point for the treatment both of permanent and induced electric polarizations. When \vec{P} is uniform —even if it's a function $\vec{P}(\vec{E}^{\text{int}})$ of the internal electric field— the last equation may be rewritten as

⁴³ M.C. Vanwormhoudt, *On the definition of macroscopic electric and magnetic fields*, Physica vol. **42**, pp. 439-446 (1969).

⁴⁴ M. Widom, *Shape-Adapted Ewald Summation*. Dekker, pp. 141-144. Reitz, pp. 81-83. Jackson pp. 115.

⁴⁵ MacMillan, pp. 27-32.

$$\vec{E}(\vec{r}) = \vec{E}^0 + k_1 \nabla \cdot \left(\nabla \cdot \left(\iiint_V \frac{1}{|\vec{r} - \vec{r}'|} d^3 r' \right) \vec{P}(\vec{E}^{\text{int}}) \right). \quad (2.5)$$

From eqs. 3.3 and A1.4, it follows that

$$\vec{E}(\vec{r}) = \vec{E}^0 - \frac{\lambda}{\epsilon_0} \mathbf{n}(\vec{r}) \cdot \vec{P}(\vec{E}^{\text{int}}) \quad \forall \vec{r}, \quad (2.6)$$

valid both inside and outside V . The depolarization tensor \mathbf{n} is defined by eq. 3.4, being uniform only inside ellipsoidal bodies, as pointed out in page 2. In order to solve this equation it is necessary to write separate expressions for the regions inside and outside the body:

$$\boxed{\begin{aligned} \vec{E}^{\text{int}} &= \vec{E}^0 - \frac{\lambda}{\epsilon_0} \mathbf{N} \cdot \vec{P}(\vec{E}^{\text{int}}) \quad \text{para } \vec{r} \in V, \\ \vec{E}^{\text{ext}}(\vec{r}) &= \vec{E}^0 - \frac{\lambda}{\epsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{P}(\vec{E}^{\text{int}}) \quad \text{para } \vec{r} \notin V, \end{aligned}} \quad (2.7)$$

where \vec{E}^{int} is the uniform internal electric field, $\vec{E}^{\text{ext}}(\vec{r})$ the external one, \mathbf{N} the internal depolarization tensor (uniform for ellipsoids) and $\mathbf{n}^{\text{ext}}(\vec{r})$ the non-uniform external one which is always a function of position \vec{r} . The eigenvalues of matrix \mathbf{N} are always positive, making the internal electric field (and the magnetic one, as will be seen later for the ferromagnetic case) smaller in magnitude than the applied one, fact that gave origin to the tensor's name. Equations 2.7 are valid both for permanent and induced polarizations, but each case is solved differently. When an ellipsoidal body is uniformly polarized—state not spontaneously acquired for the permanent case—the first equation shows that the resulting internal electric field is also uniform.

Potential eq. 2.3 may be expressed in terms of polarization densities. This requires a different use of identities A2.2 and A2.5:

$$\frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \vec{P}(\vec{r}') \cdot \nabla' \frac{1}{|\vec{r} - \vec{r}'|} = \nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \frac{\nabla' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|}, \quad (2.8)$$

where ∇' operates on \vec{r}' . Replacement of the integrand in eq. 2.3 then gives

$$\phi_p(\vec{r}) = k_1 \iiint_V \nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d^3 r' - k_1 \iiint_V \frac{\nabla' \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r'. \quad (2.9)$$

The first volume integral appears well suited for the application of divergence theorem eq. A3.8, and most textbooks apply it without further ado, replacing it

with a surface integral⁴⁶. In fact, further analysis is required because \vec{P} has a step discontinuity on the body's surface because it is finite inside and vanishes outside. That is, the derivatives of vector \vec{P} do not fulfill the standard conditions for the application of the theorem because they may be different on both sides of the body's surface⁴⁷. It turns out that this does not affect the value of the divergence in the volume integral and of the flux in the surface integral. The proof is sketched in [Appendix 3](#) (see the derivation of eq. [A3.21](#)).

It is thus obtained

$$\phi_p(\vec{r}) = k_1 \iiint_V \frac{\rho_p(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' + k_1 \iint_S \frac{\sigma_p(\vec{r}') d^2r'}{|\vec{r} - \vec{r}'|}, \quad (2.10)$$

where $\rho_p(\vec{r}) = -\nabla \cdot \vec{P}(\vec{r})$, $\sigma_p(\vec{r}) = \vec{P}(\vec{r}) \cdot \hat{s}(\vec{r})$.

where $\hat{s}(\vec{r})$ is the unit vector normal to surface S at point \vec{r} (see eq. [A7.8](#)). These charges, as real as the conduction ones, originate in the relative displacement of the baricenters of nuclear and electronic charges of atoms and molecules. The expression will later be used for solving the case of conductors. As only uniform polarizations will be considered in this book, ρ_p vanishes and the only contribution to the electric field is that of the surface density of charge σ_p . This explains the fact that inside the body the magnitude of the electric field is smaller than in the outside, as shown in [Figure 1](#).

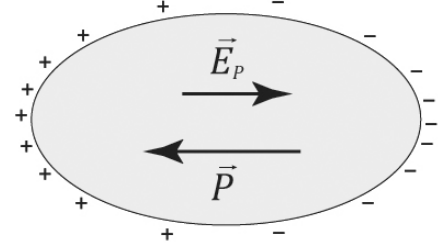


Figure 1. Origin of the depolarization effect.

The electric displacement vector $\vec{D}(\vec{r})$ is introduced in order to discriminate the polarization charges, bound to atoms and molecules, from the free conduction ones:

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \lambda \vec{P}(\vec{r}). \quad (2.11)$$

This constitutive equation for the electric case will be seldom used here, the magnetic case being quite different.

Permanent electric polarization

The necessary condition for the presence of permanent electric polarization is for some atoms or molecules of the material —ferroelectric, electret or the like below their critical temperature— to have a finite electric dipole moment in the absence of an applied electric field ($\vec{E}^0 = 0$). The state of polarization of these

⁴⁶ See, for instance: Reitz, eq. 4-12 in p. 78.

⁴⁷ J. Rey Pastor, P. Pi Calleja, and C. Trejo, *An  lisis Matem  tico* vol. 1, p. 437 (1959, 4th edition).

materials depends on the body's internal field $E^{\text{int}}(\vec{r})$, but also on other microscopic and macroscopic phenomena. The dominant ones among the former are the atomic and molecular interactions, whose magnitude may outweigh the applied field. Among the latter the most important ones are the atomic and molecular vibrations (temperature) and the existence of domains, local ordering of electric moments that tends to minimize their interaction energy. The influence of domains is a difficult topic because their size and statistical distribution are critically dependent on the material's thermal history, their evolution consisting in transitions between metaestable states⁴⁸. This leads to irreversibility (hysteresis) and a usually non-uniform polarization.

A reasonable approximation to uniform polarization may be obtained with a controlled process involving a combined variation of temperature and the application of a high field (saturation). Once a sufficiently uniform polarization has been obtained, one has to determine its value. The most common way of doing this is to suspend the body in a uniform field and measure the applied torque. The relationship between field and torque involves the depolarization tensor (see the corresponding section in page 105).

In this book no further details will be given of these topics, which are thoroughly discussed in specialized books and scientific journals. What matters here is that —whatever the method used— the necessary condition for a uniform permanent polarization, although not a sufficient one, is an ellipsoidal shape. Edges and corners, for example, are examples of shapes that preclude a state of uniform polarization even in extremely high applied fields.

When a uniform permanent polarization \vec{P} is established and its value is determined, the depolarization tensor method can be used to calculate both the internal and external electric field through eqs. 2.7. These equations, rewritten in matrix notation, read

$$\begin{aligned} \mathbf{E}^{\text{int}} &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{N} \cdot \mathbf{P}(\mathbf{E}^{\text{int}}) \quad \text{for } \vec{r} \in V, \\ \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{P}(\mathbf{E}^{\text{int}}) \quad \text{for } \vec{r} \notin V. \end{aligned} \tag{2.12}$$

The external field \mathbf{E}^{ext} cannot be determined without solving first the equation for \mathbf{E}^{int} and then replacing its value in the second of eqs. 2.12. This requires the knowledge of the function $\vec{P}(\vec{E}^{\text{int}})$, which in the isotropic case is usually taken to be Langevin's function⁴⁹. Problem 06 of chapter 5 gives a tentative graphical method for this determination. In anisotropic materials the solution requires the separate

⁴⁸ Brown (1963).

⁴⁹ Dekker, pp. 138 and 192.

determination of a number of functions determined by the single crystal symmetries (see [Appendix 5](#)).

Induced electric polarization

In the absence of permanent polarization the induced polarization is usually only a function of the applied field \vec{E}^0 , the electric susceptibility χ_e of the magnetic material⁵⁰ and the body's shape. This is so because the state of polarization of any point of the body depends on the internal electric field, the addition of the applied field and that generated by the rest of the body. The induced field should, therefore, be determined in a self-consistent way, its value at each point depending on the value at all other points. The mathematical expression of this dependence is clearly expressed by the integral eq. [2.31](#).

The applied field, the polarization and the total electric field are in general not parallel due to a combination of the anisotropy of the susceptibility (a material property) and the shape anisotropy, a geometric property characterized by the depolarization tensor, linear only in the case of ellipsoidal bodies. That is, the linearity of the relationship between the polarization and the applied field is not a mere consequence of the linearity of the field equations, as invoked by Landau and Lifschitz to justify the introduction of the body's polarizability tensor. They claim there that⁵¹ *Since all the field equations are linear, it is evident...* In fact, there is a complex relationship among the applied field components, the ensuing polarization and the resulting total field, mediated by the material's properties and the body's geometry. In the treatment made by standard texts on the theory of electromagnetism these relationships are hidden in the boundary conditions of differential equations. In the method described by this book both the anisotropy of the material and the influence of geometry are explicitly expressed by the susceptibility and depolarization tensors, while the interrelationship among fields is expressed in terms of linear systems of equations among components, clearly visible and soluble in matrix representation. A better understanding of this abstract behaviour may be obtained from a concrete illustration. Two specific examples are given to that end, the first one in [Figure 2](#), the second in the following section.

[Figure 2](#) is a storyboard of the timed sequence of interactions of two polarizable atoms subject, in vacuum, to an applied field. The same problem is next mathematically solved using matrices, in a fashion closely resembling the depolarization tensor method, to which it is an introduction.

The figure relies on the experimental fact that fields do not act instantaneously, but propagate with the velocity c of electromagnetic waves. The applied field arrives to atom 1 at time t_1 , inducing a dipole moment which acquires its full value in a later time t_2 , when atom 2 is still unpolarized. At time t_3 the applied field arrives to atom 2, but not so the one created by the dipole moment induced on atom 1. Molecule 2 acquires its initial dipole moment at time t_4 , but this value will change when the induced field of atom 1 reaches its location. Shortly after that, at

⁵⁰ Remember that this is a low field approximation, the most common but not universal case.

⁵¹ Landau and Lifchitz, p. 7.

time t_5 , the modified dipole moment of atom 2 is fully established. But the process does not end here, because atom 2 creates a new induced field that acts on atom 1, modifying again its dipole moment at time t_6 . The successive changes of polarizations and fields take a certain time to reach a value within a small but non-zero precision range. It is thus clearly seen how the values obtained are necessarily self-consistent, because the dipole moment of each atom depends on that of the other through the induced fields created by both.

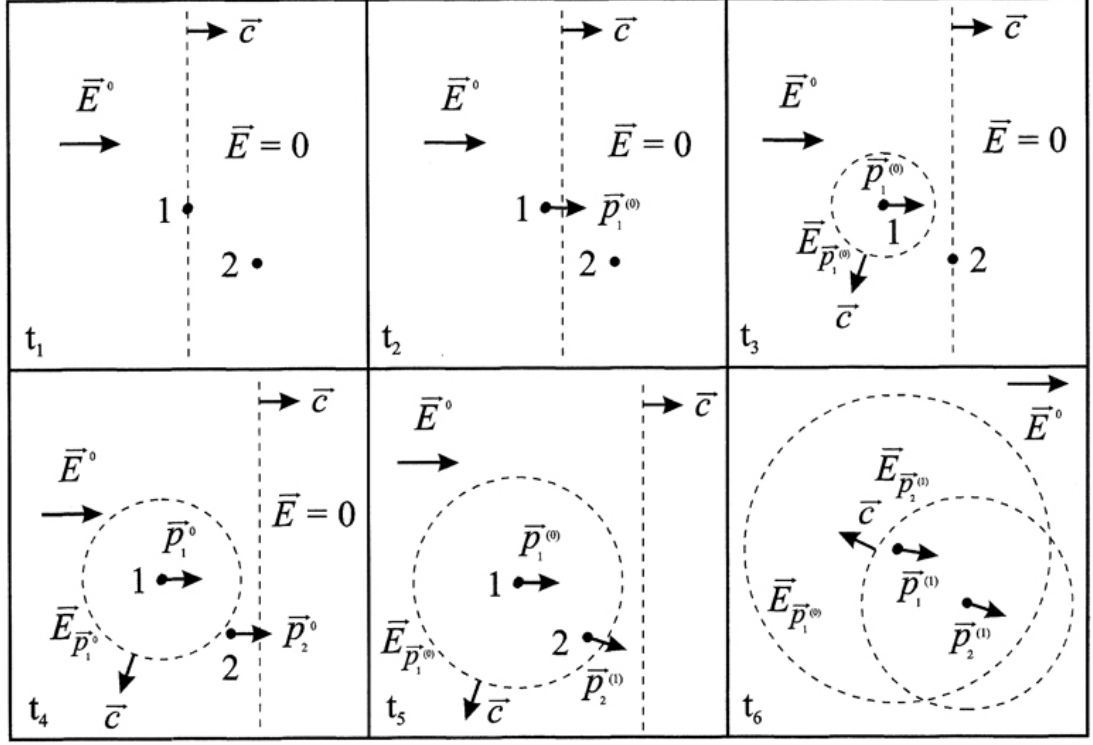


Figure 2. Autoconsistent mutual polarization of two atoms.

Induced electric polarization of two interacting atoms

In order to find the equilibrium values of the induced electric dipole moments and total field it will be assumed that both atoms are identical, isotropic and have no permanent electric moments. This happens, for instance, for closed electronic shells. The atoms, a distance d apart, are immersed in a uniform applied electric field \mathbf{E}^0 and the induced fields will be calculated using the point dipole microscopic model.

Under the influence of an applied uniform field \mathbf{E}^0 , electric dipole moments $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ are induced on the atoms at $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$. In the linear range, for an isotropic atom j its induced dipole moment is related to the microscopic electric field experienced by

$$\mathbf{p}^{(j)} = \gamma \mathbf{E}(\mathbf{r}^{(j)}), \quad (2.13)$$

where constant γ is the same for both atoms. Field $\mathbf{E}(\mathbf{r}^{(1)})$ is the addition of the applied field \mathbf{E}^0 and the field $\mathbf{E}^{(2)}(\mathbf{r}^{(1)})$ generated at $\mathbf{r}^{(1)}$ by the dipole moment of the atom at $\mathbf{r}^{(2)}$. In the rest of this section arguments of functions are always field

points and upper indices identify source points. Using eq. A4.5 and the following property of the interatomic distance⁵²

$$\vec{d}^{(1)}(\vec{r}^{(2)}) = \vec{r}^{(2)} - \vec{r}^{(1)} = \vec{d} = -\vec{d}^{(2)}(\vec{r}^{(1)}) = -(\vec{r}^{(1)} - \vec{r}^{(2)}), \quad (2.14)$$

it turns out that a matrix \mathbf{m} may be defined for both atoms, in terms of vector \vec{d} , such that

$$\mathbf{E}^{(1)}(\mathbf{r}^{(2)}) = k_1 \mathbf{m} \cdot \mathbf{p}^{(1)}, \quad \mathbf{E}^{(2)}(\mathbf{r}^{(1)}) = k_1 \mathbf{m} \cdot \mathbf{p}^{(2)}. \quad (2.15)$$

From eq. A4.5 it is obtained

$$\mathbf{m} = \frac{1}{d^5} \begin{pmatrix} 3d_x^2 - d^2 & 3d_x d_y & 3d_x d_z \\ 3d_y d_x & 3d_y^2 - d^2 & 3d_y d_z \\ 3d_z d_x & 3d_z d_y & 3d_z^2 - d^2 \end{pmatrix}, \quad (2.16)$$

where $d = |\vec{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2}$,

d_α being the components of vector \vec{d} .

From eq. 2.13, it follows that the fields and dipole moments are determined by the following system of equations:

$$\begin{aligned} \mathbf{E}(\mathbf{r}^{(1)}) &= \mathbf{E}^0 + \mathbf{E}^{(2)}(\mathbf{r}^{(1)}) = \mathbf{E}^0 + k_1 \mathbf{m} \cdot \mathbf{p}^{(2)} = \mathbf{E}^0 + k_1 \mathbf{m} \cdot \gamma \mathbf{E}(\mathbf{r}^{(2)}), \\ \mathbf{E}(\mathbf{r}^{(2)}) &= \mathbf{E}^0 + \mathbf{E}^{(1)}(\mathbf{r}^{(2)}) = \mathbf{E}^0 + k_1 \mathbf{m} \cdot \mathbf{p}^{(1)} = \mathbf{E}^0 + k_1 \mathbf{m} \cdot \gamma \mathbf{E}(\mathbf{r}^{(1)}), \end{aligned} \quad (2.17)$$

where it is clearly seen that the field on atom 1 (upper equation) explicitly depends of the field on atom 2. As the field on atom 2 explicitly depends of the field on atom 2 (lower equation), this self-consistency is expressed by the following pair of matrix equations (that is, a system of 6 linear equations):

$$\mathbf{E}(\mathbf{r}^{(1)}) = \mathbf{E}^0 + k_1 \gamma \mathbf{m} \cdot \mathbf{E}(\mathbf{r}^{(2)}), \quad \mathbf{E}(\mathbf{r}^{(2)}) = \mathbf{E}^0 + k_1 \gamma \mathbf{m} \cdot \mathbf{E}(\mathbf{r}^{(1)}). \quad (2.18)$$

As the two atoms are identical, upon its exchange the system remains unchanged. That is, $\mathbf{E}(\mathbf{r}^{(1)}) = \mathbf{E}(\mathbf{r}^{(2)})$. This symmetry argument may be verified by subtracting the second equation from the first obtaining

$$\mathbf{E}(\mathbf{r}^{(1)}) - \mathbf{E}(\mathbf{r}^{(2)}) = \mathbf{1} \cdot (\mathbf{E}(\mathbf{r}^{(1)}) - \mathbf{E}(\mathbf{r}^{(2)})) = k_1 \gamma \mathbf{m} \cdot (\mathbf{E}(\mathbf{r}^{(2)}) - \mathbf{E}(\mathbf{r}^{(1)})), \quad (2.19)$$

where $\mathbf{1}$ is the unit matrix. That is

⁵² The notation, which appears to be excessively detailed, is well suited for the calculation of lattice sums, the microscopic fields created in single crystals by ordered atoms and molecules.

$$(\mathbf{1} - k_1 \gamma \mathbf{m}) \cdot (\mathbf{E}(\mathbf{r}^{(1)}) - \mathbf{E}(\mathbf{r}^{(2)})) = 0. \quad (2.20)$$

As the left member's square matrix is non-singular (see below), the column matrix must vanish, verifying that

$$\mathbf{E}(\mathbf{r}^{(1)}) = \mathbf{E}(\mathbf{r}^{(2)}) = \mathbf{E}. \quad (2.21)$$

Therefore, the original two equations system reduces to the single matrix equation (notice the similarity with the second of eqs. 2.7)

$$\mathbf{E} = \mathbf{E}^0 + k_1 \gamma \mathbf{m} \cdot \mathbf{E}, \quad (2.22)$$

from which \mathbf{E} may be obtained, as follows:

$$(\mathbf{1} - k_1 \gamma \mathbf{m}) \cdot \mathbf{E} = \mathbf{E}^0, \quad \mathbf{E} = (\mathbf{1} - k_1 \gamma \mathbf{m})^{-1} \cdot \mathbf{E}^0. \quad (2.23)$$

Matrix $\mathbf{1} - k_1 \gamma \mathbf{m}$ inverse is easily obtained in the coordinate system where it is diagonal (its principal one), usually corresponding to the atomic system highest symmetry. In this case the highest symmetry is an arbitrary rotation around the atomic axis defined by vector \vec{d} . In fact, if one chooses a coordinate system such that both atoms are on axis z , and the origin is arbitrarily chosen to be on atom 1, it is obtained

$$d_x = d_y = 0, \quad d_z = d, \quad \mathbf{m} = \begin{pmatrix} -1/d^3 & 0 & 0 \\ 0 & -1/d^3 & 0 \\ 0 & 0 & 2/d^3 \end{pmatrix}. \quad (2.24)$$

Then

$$\begin{aligned} \mathbf{E} &= \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = (\mathbf{1} - k_1 \gamma \mathbf{m})^{-1} \cdot \mathbf{E}^0 \\ &= \begin{pmatrix} 1/1 + k_1 \gamma / d^3 & 0 & 0 \\ 0 & 1/1 + k_1 \gamma / d^3 & 0 \\ 0 & 0 & 1/1 - 2k_1 \gamma / d^3 \end{pmatrix} \cdot \begin{pmatrix} E_x^0 \\ E_y^0 \\ E_z^0 \end{pmatrix}. \end{aligned} \quad (2.25)$$

Choosing the proportionality constant in such a way that both members have the same units (see eq. A1.3), one may define the atomic electric polarizability tensor α_e by

$$\mathbf{p} = \gamma \mathbf{E} = \frac{\epsilon_0}{\lambda} \alpha_e \cdot \mathbf{E}^0, \quad \text{where} \quad \alpha_e = \frac{\lambda}{\epsilon_0} \gamma (\mathbf{1} - k_1 \gamma \mathbf{m})^{-1}. \quad (2.26)$$

The matrix \mathbf{C} relating the resultant to the applied field, $\mathbf{E} = \mathbf{C} \cdot \mathbf{E}^0$, has the typical form of the tensorial properties of uniaxial materials⁵³,

$$\mathbf{C} = \begin{pmatrix} c_{\perp} & 0 & 0 \\ 0 & c_{\perp} & 0 \\ 0 & 0 & c_{\parallel} \end{pmatrix} \quad \text{where } c_{\perp} < c_{\parallel}, \quad (2.27)$$

in the single crystal's principal coordinate system. A simple estimation illustrates that the difference in value of these two coefficients is easily measurable. $k_1\gamma$ is of order R^3 , R being the atomic radius⁵⁴. In solids the interatomic distance d is or order $2R$. Therefore, $k_1\gamma/d^3$ is about 0,1 which gives

$$\frac{k_1\gamma}{d^3} \approx 0,10, \quad c_{\perp} \approx 0,91, \quad c_{\parallel} \approx 1,25, \quad (2.28)$$

the parallel coefficient being about 37% larger than the one perpendicular to the interatomic axis.

The previous calculation corresponds to the local field of solid state physics, the microscopic electric field experienced by an atom. This field does not coincide with the macroscopic one described by Maxwell's equations, a suitable average of the former or its potential over a single crystal's unit cell or a sufficiently large number of atoms or molecules in the polycrystalline or amorphous case. An important feature of macroscopic fields, as discussed in page 6, is that any of the body's internal points is both a field and a source point. The singularities thus created did not arise in the previous treatment because point dipoles were used. A consequence of this approximation is that matrix \mathbf{m} corresponds to the exterior depolarization tensor \mathbf{n}^{ext} , as shown by its zero trace (see eq. 3.15). This suggests that it may not be feasible to obtain the internal depolarization tensor \mathbf{N} as an average of microscopic tensors like \mathbf{m} , unless significative mathematical and conceptual changes are made in the treatment. On the other hand, atoms are not geometrical points, but no concept related to the depolarization tensor seems to have been introduced for its (necessarily) quantum analysis.

Dielectrics

As in the case of permanent polarization, the starting point is eq. 2.12. In most dielectrics applied fields are significantly smaller than the effective electric fields of quantum interactions among electrons. It is then valid to keep only first order terms in the Taylor's expansion of $\vec{P}(\vec{E}(\vec{r}))$:

$$\vec{P}(\vec{r}) = \frac{\epsilon_0}{\lambda} \chi_e \cdot \vec{E}(\vec{r}). \quad (2.29)$$

⁵³ Nye, p. 23 Table 3, see [Appendix 5](#) in the present book.

⁵⁴ Dekker, p. 135. C. Kittel, *Introduction to Solid State Physics*, Wiley and Sons, p. 229, 1979.

There is no general agreement on the constant one should use for the definition of electric susceptibility. The adimensional definition given above and in the book by Stratton⁵⁵ simplifies our main formulas, while Reitz convention introduces no factor in eq. 2.29⁵⁶. The reader should carefully verify the definition used by his favourite author. With the above convention the following value of electric permeability $\boldsymbol{\varepsilon}$ is obtained,

$$\vec{D} = \varepsilon_0 \vec{E} + \lambda \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \boldsymbol{\chi}_e \cdot \vec{E} = \boldsymbol{\varepsilon} \cdot \vec{E}, \quad \text{where } \boldsymbol{\varepsilon} = \varepsilon_0 (\mathbf{1} + \boldsymbol{\chi}_e). \quad (2.30)$$

In homogeneous materials, the only ones studied in this book, the adimensional electric susceptibility $\boldsymbol{\chi}_e$ is a uniform and symmetric rank 2 tensor⁵⁷ which reduces to a scalar in the cases of non textured polycrystalline and amorphous materials. In the anisotropic case some of its components may be related, depending on the symmetries of the material's crystalline structure⁵⁸.

From eq. 2.5, the total electric field due to the application of a uniform field to a dielectric with the linear property eq. 2.29 satisfies the equation

$$\vec{E}(\vec{r}) = \vec{E}^0 + \frac{1}{4\pi} \nabla \cdot \left(\nabla \cdot \iiint_V \frac{\boldsymbol{\chi}_e \cdot \vec{E}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' \right). \quad (2.31)$$

This system of integro-differential equations univocally determines the electric field⁵⁹ and, through eq. 2.29, the polarization. In the permanent polarization case the internal field's uniformity was a consequence of a deliberately established uniform polarization. Here, on the contrary, the uniform character of the field is a consequence of eq. 2.31, from which follows that of the polarization.

A uniform internal electric field \vec{E}^{int} provides a solution to eqs. 2.31 if the following equations are satisfied (which are easily obtained in the same way as eqs. 2.7):

$$\begin{aligned} \vec{E}^{\text{int}} &= \vec{E}^0 - \mathbf{N} \cdot \boldsymbol{\chi}_e \cdot \vec{E}^{\text{int}}, \quad \vec{P} = \frac{\varepsilon_0}{\lambda} \boldsymbol{\chi}_e \cdot \vec{E}^{\text{int}} \quad \text{for } r \in V, \\ \vec{E}^{\text{ext}}(\vec{r}) &= \vec{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{P} \quad \text{for } r \notin V. \end{aligned} \quad (2.32)$$

In matrix representation it is easy to obtain \mathbf{E}^{int} from the first equation from which the values of \mathbf{P} and \mathbf{E}^{ext} immediately follow:

⁵⁵ Stratton, p. 12 eq. 8.

⁵⁶ Reitz, p. 86.

⁵⁷ Landau and Lifchitz, p. 58.

⁵⁸ Nye, p. 23 Table 3: see [Appendix 5](#) of the present book.

⁵⁹ See the initial discussion in [Solving the integro-differential equations by iteration](#).

$$\begin{aligned}
\mathbf{E}^{\text{int}} &= (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1} \cdot \mathbf{E}^0 \quad \text{for } r \in V, \\
\mathbf{P} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\chi}_e \cdot \mathbf{E}^{\text{int}} = \frac{\varepsilon_0}{\lambda} \boldsymbol{\chi}_e \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1} \cdot \mathbf{E}^0, \\
\mathbf{p} = V\mathbf{P} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\alpha}_e \cdot \mathbf{E}^0, \quad \boldsymbol{\alpha}_e = V \boldsymbol{\chi}_e \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1}, \\
\mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{P} \quad \text{for } r \notin V,
\end{aligned} \tag{2.33}$$

where \mathbf{A}^{-1} is the inverse of matrix \mathbf{A} . The relationship between the geometry of ellipsoidal bodies and electric polarization \mathbf{P} was established by Landau and Lifschitz for the isotropic case⁶⁰ through a generalization, which is not a proof, of the results obtained for the sphere and the infinitely long cylinder:

The existence of such a relationship follows from the form of the boundary conditions, as we saw above on the examples of the sphere and the cylinder.

The geometric connection, previously extended by this author to the anisotropic electric case⁶¹, is but a simple consequence of the method that allows the introduction of $\boldsymbol{\alpha}_e$, the body's electric polarizability tensor with units of volume:

$$\mathbf{p} = V\mathbf{P} = \frac{\varepsilon_0}{\lambda} \boldsymbol{\alpha}_e \cdot \mathbf{E}^0, \quad \boldsymbol{\alpha}_e = V \boldsymbol{\chi}_e \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1}. \tag{2.34}$$

This tensor was defined by Landau and Lifschitz⁶² only for conductors and magnetic materials. Notice the non-casual similarity of the formula with eq. 2.26. This tensor will be used in Chapter 5 for the calculation of the body's energy and applied torques.

Magnetization

The behaviour of ellipsoidal bodies under applied uniform magnetic fields \vec{H}^0 — the first work published by the author on the depolarization tensor method⁶³ — may be solved in a similar way as the electric case.

Basic equations

Unlike the electric case —where the treatment of polarized matter is done using the fundamental field \vec{E} — in the permanent and induced magnetic cases use is

⁶⁰ Landau and Lifchitz, p. 44.

⁶¹ Soliv  rez (2008), eq. 17.

⁶² Landau and Lifchitz, pp. 7 and 192.

⁶³ Soliv  rez (1981).

made of the auxiliar field \vec{H} . The magnetic induction \vec{B} will be used only in the case of superconductors, where use must be made of the constitutive eq. [A1.10](#),

$$\vec{B} = \mu_0 (\vec{H} + \lambda' \vec{M}). \quad (2.35)$$

The reason for using the magnetic field instead of the magnetic induction is that all equations can then be derived as for the electric case because of the existence of the following magnetic potential ϕ_M ⁶⁴:

$$\vec{H}_M(\vec{r}) = -\nabla \phi_M(\vec{r}), \quad \text{where} \quad \phi_M(\vec{r}) = \frac{\lambda'}{4\pi} \iiint_V \frac{\vec{M}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3 r'. \quad (2.36)$$

Besides, \vec{H} plays a distinguished role in the study of permanente magnets. The given equation is similar to eq. [2.1](#) of the electrostatic case. In the same fashion as for the derivation of eq. [2.5](#), one obtains, upon addition of an applied uniform field, that the total magnetic field is

$$\boxed{\vec{H}(\vec{r}) = \vec{H}^0 + \frac{\lambda'}{4\pi} \nabla \cdot \left(\nabla \cdot \left(\iiint_V \frac{1}{|\vec{r} - \vec{r}'|} d^3 r' \right) \vec{M} \right)}. \quad (2.37)$$

This equation, the basis of the analysis of permanent and induced polarizations of ellipsoidal bodies, is valid in all space independently of the source and spatial dependence of $\vec{M}(\vec{r})$. When \vec{M} is uniform, case in which no argument is written, the previous equation becomes

$$\vec{H}(\vec{r}) = \vec{H}^0 + \frac{\lambda'}{4\pi} \nabla \cdot \left(\nabla \cdot \left(\iiint_V \frac{1}{|\vec{r} - \vec{r}'|} d^3 r' \right) \vec{M} \right). \quad (2.38)$$

From eq. [3.3](#) it follows that

$$\vec{H}(\vec{r}) = \vec{H}^0 - \lambda' \mathbf{n}(\vec{r}) \cdot \vec{M}(\vec{H}) \quad \forall \vec{r}. \quad (2.39)$$

As previously done for the electric case, it is more convenient to rewrite this equation in order to explicitly separate the body's internal and external regions:

$$\begin{aligned} \vec{H}^{\text{int}} &= \vec{H}^0 - \lambda' \mathbf{N} \cdot \vec{M}(\vec{H}^{\text{int}}) \quad \text{for } \vec{r} \in V, \\ \vec{H}^{\text{ext}}(\vec{r}) &= \vec{H}^0 - \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{M}(\vec{H}^{\text{int}}) \quad \text{for } \vec{r} \notin V. \end{aligned} \quad (2.40)$$

These equations are valid both for permanent and induced polarizations, but they are solved differently in each case. Therefore, if an ellipsoidal body is uniformly

⁶⁴ A detailed reading of pp. 189-194 of Reitz's book is recommended.

magnetized (which requires a nontrivial preparation), the resulting internal magnetic field is also uniform.

For the treatment of superconductors it is convenient to derive the magnetic field from the surface density of current, case that bears no resemblance to that of a surface density of charge (conductors). To that end it is convenient to use the magnetic induction, derivable from the following vector potential⁶⁵:

$$\vec{A}_M(\vec{r}) = \frac{\mu_0 \lambda'}{4\pi} \iiint_V \frac{\vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3 r', \quad \vec{B}(\vec{r}) = \nabla \times \vec{A}_M(\vec{r}), \quad (2.41)$$

The integrand may now be transformed using eqs. A2.2 and A2.8, obtaining

$$\begin{aligned} \vec{A}_M(\vec{r}) &= \frac{\mu_0 \lambda'}{4\pi} \iiint_V \frac{\vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3 r' = \frac{\mu_0 \lambda'}{4\pi} \iiint_V \vec{M}(\vec{r}') \times \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 r' \\ &= \frac{\mu_0 \lambda'}{4\pi} \iiint_V \frac{\nabla' \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' - \frac{\mu_0 \lambda'}{4\pi} \iiint_V \nabla' \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d^3 r'. \end{aligned} \quad (2.42)$$

The last volume integral may be transformed into a surface one by the curl theorem eq. A3.10, where due attention should be paid to the step discontinuity through the body's surface (see eq. A3.22). It is thus obtained

$$\begin{aligned} \vec{A}_M(\vec{r}) &= \frac{\mu_0 \lambda'}{4\pi} \iiint_V \frac{\vec{J}_M(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' + \frac{\mu_0 \lambda'}{4\pi} \oint_S \frac{\vec{K}_M(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2 r', \\ \text{where } \vec{J}_M(\vec{r}) &= \nabla \times \vec{M}(\vec{r}), \quad \vec{K}_M(\vec{r}) = \vec{M}(\vec{r}) \times \hat{s}(\vec{r}), \end{aligned} \quad (2.43)$$

where $\hat{s}(\vec{r})$ is the unit vector normal to surface S at point \vec{r} (see eq. A7.8), \vec{J}_M is the volume density of magnetization current and \vec{K}_M the surface density.

The previous equations show how magnetization may originate from currents localized in the interior and on the surface of the body. The validity of this interpretation for superconductors is discussed in the relevant section and will be also used to show that they behave as perfect diamagnetic materials.

Permanent magnetization

The study of spontaneous magnetization and the phenomena of saturation and hysteresis preceded that of ferroelectric materials, which took their name from ferromagnets. As discussed in the ferroelectric case, a magnetization is in general not only a function of the applied magnetic field, the material and its shape, but also of its own buildup process.

⁶⁵ Reitz, p. 190 eq. 9-11.

The resulting equations 2.40 are similar to those obtained in page 18 for permanent electric polarization. Writing them in matrix representation it is obtained

$$\begin{aligned} \mathbf{H}^{\text{int}} &= \mathbf{H}^0 - \lambda' \mathbf{N} \cdot \mathbf{M}(\mathbf{H}^{\text{int}}) \quad \text{for } \vec{r} \in V, \\ \mathbf{H}^{\text{ext}}(\vec{r}) &= \mathbf{H}^0 - \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{M}(\mathbf{H}^{\text{int}}) \quad \text{for } \vec{r} \notin V. \end{aligned} \quad (2.44)$$

The scalar and isotropic version of the first equation, with $M(H)$ proportional to Brillouin's function⁶⁶, is customarily used in the study of ferromagnetism⁶⁷. The applied field, the magnetization and the total field have in general different directions. The same thing happens in isotropic materials for all but spherical bodies, due to the geometrical anisotropy introduced by the depolarization tensor \mathbf{n} (see Problem 06).

Induced magnetization

Unlike the electric case, in the magnetic one there are two different types of induced polarization: diamagnetic and paramagnetic. Diamagnetism is a manifestation of Lenz' Law, a change in the orbital circulation of electrons that, as happens with any current, generates a field that opposes the applied one. Paramagnetism stems from the orientation of the permanent magnetic moments of atoms without closed electronic shells. Both types are well described in the linear range by

$$\vec{M}(\vec{r}) = \frac{1}{\lambda'} \chi_m \cdot \vec{H}(\vec{r}), \quad (2.45)$$

where χ_m is the magnetic susceptibility, a symmetric rank two tensor⁶⁸. As in the electric case, there is no general agreement on the definition of magnetic susceptibility, and the reader should check the one used by his favourite author, introducing the necessary additional constants. The definition given by eq. 2.45 simplifies the formulas eqs. 2.49. The magnetic susceptibility is a negative definite tensor in the diamagnetic case and a positive definite one in the paramagnetic case, uniform for all the homogeneous materials studied in this book and a scalar for polycrystalline non-textured and amorphous materials. In the single crystal anisotropic case its components have relationships determined by the symmetry of the material's lattice structure⁶⁹.

⁶⁶ Dekker, pp. 454-455.

⁶⁷ Dekker, p. 466.

⁶⁸ Landau and Lifchitz, p. 58.

⁶⁹ Nye, p. 23 Table 3. See Appendix 5 in the present book.

For induced magnetizations eq. 2.37 reduces to

$$\vec{H}(\vec{r}) = \vec{H}^0 + \frac{1}{4\pi} \nabla \left(\nabla \cdot \iiint_V \frac{\vec{\chi}_m \cdot \vec{H}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' \right), \quad (2.46)$$

system of integro-differential equations that univocally determines the magnetic field. When \vec{H}^{int} is uniform inside the body —case in which the spatial argument is omitted— the previous equation reduces to

$$\vec{H}^{\text{int}} = \vec{H}^0 + \frac{1}{4\pi} \nabla \left(\nabla \cdot \iiint_V \frac{\vec{\chi}_m \cdot \vec{H}^{\text{int}}}{|\vec{r} - \vec{r}'|} d^3r' \right). \quad (2.47)$$

From eq. 3.3 it is then obtained:

$$\begin{aligned} \vec{H}^{\text{int}} &= \vec{H}^0 - \mathbf{N} \cdot \vec{\chi}_m \cdot \vec{H}^{\text{int}}, \quad \vec{H}^{\text{int}} = (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1} \cdot \vec{H}^0 \quad \text{for } \vec{r} \in V, \\ \vec{H}^{\text{ext}}(\vec{r}) &= \vec{H}^0 - \lambda' \mathbf{n}(\vec{r}) \cdot \vec{M} \quad \text{for } \vec{r} \notin V, \\ \vec{M} &= \frac{1}{\lambda'} \vec{\chi}_m \cdot \vec{H}^{\text{int}} = \frac{1}{\lambda'} \vec{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1} \cdot \vec{H}^0, \\ \vec{m} = V \vec{M} &= \frac{1}{\lambda'} \vec{\alpha}_m \cdot \vec{H}^0, \quad \vec{\alpha}_m = V \vec{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1}. \end{aligned} \quad (2.48)$$

\vec{m} is the magnetic dipole moment and $\vec{\alpha}_m$ is the ellipsoidal body's magnetic polarizability tensor.

In matrix representation the previous equations are written

$$\begin{aligned} \mathbf{H}^{\text{int}} &= (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1} \cdot \mathbf{H}^0 \quad \text{for } \vec{r} \in V, \\ \mathbf{H}^{\text{ext}}(\vec{r}) &= \mathbf{H}^0 - \lambda' \mathbf{n}(\vec{r}) \cdot \mathbf{M} \quad \text{for } \vec{r} \notin V, \\ \text{where } \mathbf{M} &= \frac{1}{\lambda'} \vec{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1} \cdot \mathbf{H}^0, \\ \mathbf{m} = V \mathbf{M} &= \frac{1}{\lambda'} \vec{\alpha}_m \cdot \mathbf{H}^0, \quad \vec{\alpha}_m = V \vec{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \vec{\chi}_m)^{-1}. \end{aligned} \quad (2.49)$$

Conductors

If a conducting body with zero net charge is placed in an applied uniform field \vec{E}^0 , a surface density of charge σ is induced such that, in the equilibrium state, the internal field vanishes. The cancellation is not instantaneous but the culmination of a dynamic process in which the body's surface electric charges are redistributed. Electric forces alone do not suffice, as shown by Earnshaw's Theorem⁷⁰: electrons move in order to be as far away as possible from each other, being stopped only by

⁷⁰ Stratton, p. 116. The theorem establishes that no charge may be in stable equilibrium under the effect of only a macroscopic electric field.

the atomic forces that prevent them from escaping from the body at normal temperatures (at high temperatures thermoionic emission —Edison Effect— takes place⁷¹). The vanishing of the internal electric field —valid for conducting bodies of any shape, not only ellipsoids— is, therefore, an additional condition to the laws of electrostatics, not a consequence of them:

$$\boxed{\vec{E}^{\text{int}} = 0.} \quad (2.50)$$

The total electric field may be written in terms of the applied one and the resulting —but yet unknown— surface density of charge σ as⁷²

$$\vec{E}(\vec{r}) = \vec{E}^0 - k_1 \nabla \oint_S \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2 r', \quad (2.51)$$

where $\sigma(\vec{r})$ and $\vec{E}(\vec{r})$ are mutually determined. This equation may be written solely in terms of the electric field by using the step discontinuity through the body's surface eq. 3.35 (read the section [Surface step discontinuity](#)):

$$k_1 \sigma(\vec{r}) = \frac{1}{4\pi} (\vec{E}^+ - \vec{E}^-) \cdot \hat{s}(\vec{r}), \quad \text{where } \vec{E}^- = 0, \quad (2.52)$$

and $\hat{s}(\vec{r})$ is the unit vector normal to the ellipsoid's surface at point \vec{r} (see eq. A7.8). Therefore

$$\vec{E}(\vec{r}) = \vec{E}^0 - \frac{1}{4\pi} \nabla \oint_S \frac{\vec{E}^+(\vec{r}') \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}, \quad (2.53)$$

a system of integro-differential equations that fully determines the electric field. The use of the notation \vec{E}^+ stresses the step discontinuity through the body's surface, crucial for the correct application of the integral theorems of vector calculus.

Inside the body $\vec{E}(\vec{r}) = 0$, equation that can be satisfied only if the second term of the first member is a constant. This requirement is satisfied if \vec{E}^+ is uniform, case in which eq. 3.29 gives

$$\frac{1}{4\pi} \nabla \oint_S \frac{\vec{E}^+ \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} = \mathbf{N} \cdot \vec{E}^+ = \vec{E}^0, \quad (2.54)$$

where \mathbf{N} is the body's internal depolarization tensor. This equation may be used to obtain \vec{E}^+ whenever \mathbf{N} has an inverse, which is true for the case of nonvanishing

⁷¹ Dekker, p. 220.

⁷² Reitz, p. 31.

eigenvalues. If one or two of the eigenvalues are zero, a reduced inverse may be defined which solves the problem, the links being given at page 60.

The external field is then obtained as follows:

$$\begin{aligned}\vec{E}^{\text{int}} &= \vec{E}^0 - \mathbf{N} \cdot \vec{E}^+ = 0, \quad \vec{E}^+ = \mathbf{N}^{-1} \cdot \vec{E}^0 \quad \text{for } \vec{r} \in V, \\ \vec{E}^{\text{ext}}(\vec{r}) &= \vec{E}^0 - \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{E}^+ \quad \text{for } \vec{r} \notin V.\end{aligned}\tag{2.55}$$

The electric dipole moment \vec{p} of the ellipsoidal conductor is

$$\vec{p} = \oint\!\!\!\oint_S \sigma(\vec{r}') \vec{r}' d^2r',\tag{2.56}$$

where, from eq. 2.52, the surface charge density is

$$\sigma(\vec{r}) = \frac{\epsilon_0}{\lambda} \hat{s}(\vec{r}) \cdot \vec{E}^+ = \frac{\epsilon_0}{\lambda} \hat{s}(\vec{r}) \cdot \mathbf{N}^{-1} \cdot \vec{E}^0.\tag{2.57}$$

and $\hat{s}(\vec{r})$ is the outgoing unit vector normal to the body's surface at point \vec{r} .

Therefore

$$\vec{p} = \frac{\epsilon_0}{\lambda} \left(\oint\!\!\!\oint_S \vec{r}' \hat{s}(\vec{r}') d^2r' \right) \cdot \mathbf{N}^{-1} \cdot \vec{E}^0 = \frac{\epsilon_0}{\lambda} \left(\oint\!\!\!\oint_S \vec{r}' d^2\vec{r}' \right) \cdot \mathbf{N}^{-1} \cdot \vec{E}^0.\tag{2.58}$$

The integral inside the parenthesis may be evaluated by components using the gradient theorem eq. A3.9, as follows:

$$\begin{aligned}\oint\!\!\!\oint_S \left(\sum_{\alpha} \hat{x}_{\alpha} x_{\alpha} \right) d^2\vec{r} &= \sum_{\alpha} \hat{x}_{\alpha} \left(\oint\!\!\!\oint_S x_{\alpha} d^2\vec{r} \right) = \\ \sum_{\alpha} \hat{x}_{\alpha} \left(\iiint_V (\nabla x_{\alpha}) d^3r \right) &= V \sum_{\alpha} \hat{x}_{\alpha} \hat{x}_{\alpha} = V \hat{\mathbf{1}}.\end{aligned}\tag{2.59}$$

where $\hat{\mathbf{1}}$ is the unit dyadic, the one that leaves unchanged any vector or dyadic. Therefore

$$\vec{p} = \frac{\epsilon_0}{\lambda} \boldsymbol{\alpha}_e \cdot \vec{E}^0, \quad \text{where } \boldsymbol{\alpha}_e = V \mathbf{N}^{-1},\tag{2.60}$$

$\boldsymbol{\alpha}_e$ being the polarizability tensor of the conducting body.

Although it will not be used in this book (but see problem Problem 31), it should be noticed that the surface charge distribution $\sigma(\vec{r})$ of a conducting ellipsoid with net charge Q may be expressed —in the absence of applied field— in terms of the central distance of that point (see p. 172) as follows⁷³:

⁷³ Stratton, p. 209 eq. 12.

$$\sigma(\vec{r}) = \frac{d(\vec{r})}{4\pi abc} Q, \quad (2.61)$$

expression that should be compared with the density induced by a uniform field, eq. 3.32. From there it follows that

$$\oint_S d(\vec{r}) d^2r = 4\pi abc. \quad (2.62)$$

In matrix representation one gets

$$\begin{aligned} \mathbf{E}^{\text{int}} &= \mathbf{E}^0 - \mathbf{N} \cdot \mathbf{E}^+ = 0, \quad \mathbf{E}^+ = \mathbf{N}^{-1} \cdot \mathbf{E}^0 \quad \text{for } \vec{r} \in V, \\ \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{E}^+ \quad \text{for } \vec{r} \notin V \\ \mathbf{p} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\alpha}_c \cdot \mathbf{E}^0, \quad \text{where } \boldsymbol{\alpha}_c = V \mathbf{N}^{-1}. \end{aligned} \quad (2.63)$$

The value of the dipole moment is the same obtained with the polarization model eq. 6.29.

Except for the vanishing internal field, the equations for a conducting ellipsoid are similar to those of a dielectric, where vector \vec{E}^+ corresponds to the uniform polarization \vec{P} (see eq. 2.10). This suggests that the treatment for both types of materials may be unified by the introduction of a suitable equivalent polarization for conductors, a topic discussed next.

Equivalent polarization

The behaviour of electrons in conductors may be considered as an extreme case of polarization. In dielectrics the displacement of electrons under the effect of an applied field is strictly limited by the quantum energies binding them to the nuclei, which renders it microscopic, a fraction of the atomic or molecular diameter⁷⁴. In conductors, as explained by band theory⁷⁵, electrons can easily migrate from one atom to another. This may be interpreted as a macroscopic polarization, that of a material with infinite electric susceptibility, as will be shown below.

As follows from equations 2.10, in the region where a polarization is uniform the volume density of charge vanishes and its only effect comes from the step discontinuity through the body's surface. One may, then, define the following electric potential generated by an equivalent polarization \vec{P}_{eq} ⁷⁶:

⁷⁴ Dekker, p. 135.

⁷⁵ Dekker, chapter 10 pp. 238-274.

⁷⁶ Solivérez (2008), p. 207.

$$\phi_{\text{eq}}(\vec{r}) = k_1 \oint_S \left(\frac{\sigma_{\text{eq}}(\vec{r}') d^2 r'}{|\vec{r} - \vec{r}'|} \right) = k_1 \oint_S \left(\frac{\vec{P}_{\text{eq}}(\vec{r}') \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} \right). \quad (2.64)$$

This potential gives rise to the electric field

$$\vec{E}(\vec{r}) = \vec{E}^0 - k_1 \nabla \oint_S \frac{\vec{P}_{\text{eq}} \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}, \quad (2.65)$$

which coincides with eq. 2.53 if

$$\vec{E}^+ = 4\pi k_1 \vec{P}_{\text{eq}} = \frac{\lambda}{\epsilon_0} \vec{P}_{\text{eq}}, \quad (2.66)$$

where eq. A1.4 was used. The case of conductors may now be solved in the same fashion as that of dielectrics if \vec{P}_{eq} is given a value such that the internal field vanishes. This value, as shown in Problem 08 at page 127, corresponds to the polarization induced in a material with infinite electric susceptibility.

Superconductors

If a magnetic induction \vec{B}^0 is applied to a superconducting body —in an appropriate range of temperature and field intensity— surface currents are generated such that the resulting internal value of \vec{B} vanishes. This Meissner effect and the cancellation of the material's resistivity are the most notorious consequences of a complex microscopic phenomena explained by the BCS theory (initials of Bardeen, Cooper and Schrieffer) and phenomenologically described by the London model. The latter makes use of currents confined within a finite but very small distance of the body's surface⁷⁷. From the macroscopic point of view these currents may be characterized as zero thickness surface currents. The depolarization tensor method may then be used to macroscopically characterize surface currents in superconducting ellipsoidal bodies in a uniform magnetic induction \vec{B} ⁷⁸. As in the case of conductors, an additional condition, the Meissner effect, is required:

$$\boxed{\vec{B}(\vec{r}) = \mu_0 (\vec{H}(\vec{r}) + \lambda' \vec{M}(\vec{r})) = 0 \text{ for } \vec{r} \in V.} \quad (2.67)$$

This condition —the magnetic analogous of the infinite electric susceptibility of conductors— may be interpreted as a perfect diamagnetism that cancels any internal magnetic field \vec{H}^{int} . Alternately, one may assume that \vec{H} and \vec{M} vanish

⁷⁷ Reitz, pp. 325-328.

⁷⁸ Reitz, p. 319.

inside the body and —the resistivity being zero— that the Meissner effect is the consequence of a perdurable surface conduction current. Both models are used to analyse the Meissner effect. Although they are mutually incompatible, the coherent use of any of them fully describes the effect from the electromagnetic point of view⁷⁹. The depolarization tensor method will, therefore, be applied to these two models, showing that both lead to exactly the same results.

Magnetization model

Equations 2.49 describing the behaviour of magnetizable ellipsoidal bodies in a uniform field \vec{B}^0 may be rewritten as follows:

$$\begin{aligned}\vec{B}^{\text{int}} &= \mu_0 (\vec{H}^{\text{int}} + \lambda' \vec{M}), \quad \vec{H}^{\text{int}} = \frac{\vec{B}^0}{\mu_0} - \lambda' \mathbf{N} \cdot \vec{M}, \quad \text{for } \vec{r} \in V, \\ \vec{B}^{\text{ext}}(\vec{r}) &= \vec{B}^0 - \mu_0 \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{M} \quad \text{for } \vec{r} \notin V.\end{aligned}\tag{2.68}$$

The value of \vec{M} may be obtained from condition 2.67,

$$\vec{B}^{\text{int}} = \mu_0 (\vec{H}^{\text{int}} + \lambda' \vec{M}) = 0, \quad \vec{M} = -\frac{1}{\lambda'} \vec{H}^{\text{int}},\tag{2.69}$$

an equivalent magnetic polarization that corresponds to a perfect diamagnetic susceptibility, as discussed in Problem 15. Replacing this value in the second of equs. 2.66, it is obtained

$$\vec{H}^{\text{int}} = \frac{\vec{B}^0}{\mu_0} - \lambda' \mathbf{N} \cdot \vec{M} = -\lambda' \vec{M}, \quad \vec{B}^0 = \mu_0 (\lambda' \mathbf{N} \cdot \vec{M} - \lambda' \vec{M}) = \mu_0 \lambda' (\mathbf{N} - \mathbf{1}) \cdot \vec{M}.\tag{2.70}$$

Rewriting the last member so that the tensor is positive definite (has an inverse)

$$\vec{M} = -\frac{1}{\mu_0 \lambda'} (\mathbf{1} - \mathbf{N})^{-1} \cdot \vec{B}^0.\tag{2.71}$$

In this interpretation the Meissner effect is the consequence of a magnetization that cancels the internal magnetic field \vec{H}^{int} . This magnetization gives origin to a magnetic moment \vec{m} and an external magnetic induction, both functions of the applied field:

$$\begin{aligned}\vec{m} &= V \cdot \vec{M} = -\frac{V}{\mu_0 \lambda'} (\mathbf{1} - \mathbf{N})^{-1} \cdot \vec{B}^0 = \frac{1}{\mu_0 \lambda'} \alpha_s \cdot \vec{B}^0, \quad \alpha_s = -V (\mathbf{1} - \mathbf{N})^{-1} \\ \vec{B}^{\text{ext}}(\vec{r}) &= \vec{B}^0 - \mu_0 \lambda' \mathbf{n}(\vec{r}) \cdot \vec{M} = \vec{B}^0 + \mathbf{n}(\vec{r}) \cdot (\mathbf{1} - \mathbf{N})^{-1} \cdot \vec{B}^0.\end{aligned}\tag{2.72}$$

where α_s is the body's polarizability tensor of the ellipsoidal superconductor.

⁷⁹ Reitz, chapter 15 pp. 318-333.

In matrix notation

$$\begin{aligned}
 \mathbf{B}^{\text{int}} = \mu_0 (\mathbf{H}^{\text{int}} + \lambda' \mathbf{M}) = 0, \quad \mathbf{M} = -\frac{1}{\mu_0 \lambda'} (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{B}^0, \quad \text{for } \vec{r} \notin V \\
 \mathbf{B}^{\text{ext}}(\vec{r}) = \mathbf{B}^0 - \mu_0 \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{M} \quad \text{for } \vec{r} \notin V, \\
 \text{where } \mathbf{m} = V \cdot \mathbf{M} = \frac{1}{\mu_0 \lambda'} \boldsymbol{\alpha}_s \cdot \mathbf{B}^0, \quad \boldsymbol{\alpha}_s = -V (\mathbf{1} - \mathbf{N})^{-1}.
 \end{aligned} \tag{2.73}$$

The same result may be derived from eq. 2.43, the magnetization surface current discussed in Problem 17. In Problem 14 our equations are compared with those given by Reitz (eq. 15-6) for the case of a superconducting sphere.

Surface conduction current model

In this model of superconductivity the Meissner effect is derived from a surface conduction current. The difference between a conduction current—that of electric circuits—from a magnetization current is that the latter is bound to single atoms or molecules, so that no electrons are transferred between them⁸⁰. Magnetization currents are as real as conduction currents, but only the former are feasible in insulating materials.

The total magnetic induction is the addition of the applied one, \vec{B}^0 , and that generated by a surface conduction current \vec{K} as given by eq. 2.43:

$$\begin{aligned}
 \vec{B}(\vec{r}) = \vec{B}^0 + \vec{B}_K(\vec{r}), \quad \vec{B}_K(\vec{r}) = \nabla \times \vec{A}(\vec{r}), \\
 \vec{A}(\vec{r}) = \frac{\mu_0 \lambda'}{4\pi} \oint_S \frac{\vec{K}_M(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2 r', \quad \vec{K}_M(\vec{r}) = \vec{M}(\vec{r}) \times \hat{s}(\vec{r}).
 \end{aligned} \tag{2.74}$$

where $\vec{K}(\vec{r})$ is a vector field such that Meissner's effect takes place. Its value can be found by noting that any surface current \vec{K} gives rise to a magnetic's induction step discontinuity through the body's surface⁸¹,

$$\hat{s}(\vec{r}) \times (\vec{B}^+(\vec{r}) - \vec{B}^-(\vec{r})) = \hat{s}(\vec{r}) \times \vec{B}^+(\vec{r}) = \mu_0 \lambda' k_3 \vec{K}(\vec{r}) \quad \text{as } \vec{B}^-(\vec{r}) = 0, \tag{2.75}$$

where $\hat{s}(\vec{r})$ is the outgoing unit vector normal to the ellipsoid's surface S at point \vec{r} (see eq. A7.8) and the Meissner condition $\vec{B}^-(\vec{r}) = 0$ has been used. Thus, taking into account eqs. A1.11 and A7.8, the appropriate surface current is given by

⁸⁰ See, for instance, Reitz, p. 188 Figure 9-2.

⁸¹ Stratton, p. 246 eq. 7.

$$\vec{K}(\vec{r}) = \frac{k_3}{4\pi k_2} \hat{s}(\vec{r}) \times \vec{B}^+(\vec{r}), \quad \text{where} \quad \hat{s}(\vec{r}) = \frac{\frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \quad (2.76)$$

Replacement of this value in the expression of the total induction gives

$$\vec{B}(\vec{r}) = \vec{B}^0 - \frac{1}{4\pi} \nabla \times \oint_S \frac{\vec{B}^+(\vec{r}') \times d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}, \quad (2.77)$$

the integro-differential equation that determines the magnetic induction for ellipsoidal superconductor. Using as an Ansatz a uniform value \vec{B}^+ , one obtains for the second term

$$\begin{aligned} \vec{B}_K(\vec{r}) &= -\frac{1}{4\pi} \nabla \times \oint_S \frac{\vec{B}^+ \times d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} = -\frac{1}{4\pi} \nabla \times \left(\vec{B}^+ \times \oint_S \frac{d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} \right) \\ &= -\frac{1}{4\pi} \nabla \times \left(\vec{B}^+ \times \iiint_V \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 r' \right) = -\frac{1}{4\pi} \nabla \times \left(\nabla \iiint_V \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 r' \times \vec{B}^+ \right), \end{aligned} \quad (2.78)$$

where the curl theorem eq. A3.9 has been used. The last term—which for the sake of concision is written in terms of $f(\vec{r})$ (eqs. 3.8 and 3.4)—may be transformed according to the identity eq. A2.9:

$$\begin{aligned} \vec{B}_K(\vec{r}) &= \nabla \times (\nabla f(\vec{r}) \times \vec{B}^+) \\ &= (\vec{B}^+ \cdot \nabla) \nabla f(\vec{r}) - (\nabla f(\vec{r}) \cdot \nabla) \vec{B}^+ + \nabla f(\vec{r}) (\nabla \cdot \vec{B}^+) - \vec{B}^+ (\nabla \cdot \nabla f(\vec{r})) \\ &= (\vec{B}^+ \cdot \nabla) \nabla f(\vec{r}) - \vec{B}^+ \Delta f(\vec{r}), \end{aligned} \quad (2.79)$$

The expansion follows from the uniform character of \vec{B}^+ and the property eq. A2.3 of the Laplacian. Using eqs. 3.3 and A3.7 the last term may be rewritten as

$$\begin{aligned} \vec{B}_K(\vec{r}) &= -\frac{1}{4\pi} (\vec{B}^+ \cdot \nabla) \nabla \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \vec{B}^+ \Delta \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} \\ &= \begin{cases} \vec{N} \cdot \vec{B}^+ - \vec{B}^+ & \text{for } \vec{r} \in V \\ \vec{n}^{\text{ext}}(\vec{r}) \cdot \vec{B}^+ & \text{for } \vec{r} \notin V \end{cases}. \end{aligned} \quad (2.80)$$

Therefore

$$\vec{B}(\vec{r}) = \vec{B}^0 + \vec{B}_K(\vec{r}) = \begin{cases} \vec{B}^0 - (\vec{1} - \vec{N}) \cdot \vec{B}^+ & \text{for } \vec{r} \in V \\ \vec{B}^0 + \vec{n}^{\text{ext}}(\vec{r}) \cdot \vec{B}^+ & \text{for } \vec{r} \notin V \end{cases}. \quad (2.81)$$

where \vec{B}^+ is fully determined by the Meissner condition

$$\vec{B}^0 - (\mathbf{1} - \mathbf{N}) \cdot \vec{B}^+ = 0, \quad \vec{B}^+ = (\mathbf{1} - \mathbf{N})^{-1} \cdot \vec{B}^0. \quad (2.82)$$

As shown in [Problem 16](#) of page 132, the magnetic moment generated by this density \vec{K} of surface conduction current is the same as that obtained from the magnetization model eq. 2.73,

$$\mathbf{m} = -\frac{1}{\mu_0 \lambda} V \vec{B}^+ = \frac{1}{\mu_0 \lambda} \alpha_s \cdot \mathbf{B}^0, \quad \alpha_s = -V (\mathbf{1} - \mathbf{N})^{-1}. \quad (2.83)$$

In matrix representation,

$$\begin{aligned} \mathbf{B}^{\text{int}} = \mathbf{H}^{\text{int}} = \mathbf{M} = 0, \quad \mathbf{B}^+ &= (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{B}^0 \\ \mathbf{B}^{\text{ext}}(\vec{r}) &= \mathbf{B}^0 + \mathbf{n}(\vec{r}) \cdot \mathbf{B}^+, \\ \mathbf{m} &= \frac{1}{\mu_0 \lambda} \alpha_s \cdot \mathbf{B}^0, \quad \alpha_s = -V (\mathbf{1} - \mathbf{N})^{-1}, \\ \vec{K}(\vec{r}) &= \frac{k_3}{4\pi k_2} \hat{s}(\vec{r}) \times \vec{B}^+, \quad \text{where } \hat{s}(\vec{r}) = \frac{\frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \end{aligned} \quad (2.84)$$

Apart from the difference in electron's localization, the only difference between the magnetization and the surface current models of superconductivity is the assignement of the internal values of \vec{H} and \vec{M} .

Summary of integro-differential equations

Material	Equation	Eq.
dielectrics	$\vec{E}(\vec{r}) = \vec{E}^0 + \frac{1}{4\pi} \nabla \cdot \left(\nabla \cdot \iiint_V \frac{\chi_e \cdot \vec{E}(\vec{r}')}{ \vec{r} - \vec{r}' } d^3 r' \right).$	2.31
diamagnets paramagnets	$\vec{H}(\vec{r}) = \vec{H}^0 + \frac{1}{4\pi} \nabla \cdot \left(\nabla \cdot \iiint_V \frac{\chi_m \cdot \vec{H}(\vec{r}')}{ \vec{r} - \vec{r}' } d^3 r' \right),$	2.46
conductors	$\vec{E}^0 - \frac{1}{4\pi} \nabla \cdot \iiint_s \frac{\vec{E}^+(\vec{r}') \cdot d^2 \vec{r}'}{ \vec{r} - \vec{r}' } = 0 \quad \text{for } \vec{r} \in V.$	2.53
superconductors	$\vec{B}(\vec{r}) = \vec{B}^0 - \frac{1}{4\pi} \nabla \times \iiint_s \frac{\vec{B}^+(\vec{r}') \times d^2 \vec{r}'}{ \vec{r} - \vec{r}' },$	2.77

Table 1. Integro-differential equations derived in this chapter.

Summary of induced polarization equations

Dielectrics:

$$\begin{aligned}
 \mathbf{E}^{\text{dep}} &= -\mathbf{N} \cdot \boldsymbol{\chi}_e \cdot \mathbf{E}^{\text{int}}, \\
 \mathbf{E}^{\text{int}} &= (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1} \mathbf{E}^0 \quad \text{for } r \in V, \\
 \mathbf{P} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\chi}_e \cdot \mathbf{E}^{\text{int}} = \frac{\varepsilon_0}{\lambda} \boldsymbol{\chi}_e \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1} \mathbf{E}^0, \\
 \mathbf{p} = V \mathbf{P} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\alpha}_e \cdot \mathbf{E}^0, \quad \boldsymbol{\alpha}_e = V \boldsymbol{\chi}_e \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_e)^{-1}, \\
 \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{P} \quad \text{for } r \notin V.
 \end{aligned} \tag{2.33}$$

Magnetic materials:

$$\begin{aligned}
 \mathbf{H}^{\text{dep}} &= -\mathbf{N} \cdot \boldsymbol{\chi}_m \cdot \mathbf{H}^{\text{int}} \\
 \mathbf{H}^{\text{int}} &= (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_m)^{-1} \cdot \mathbf{H}^0 \quad \text{for } \vec{r} \in V, \\
 \mathbf{M} &= \frac{1}{\lambda'} \boldsymbol{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_m)^{-1} \cdot \mathbf{H}^0, \\
 \mathbf{m} = V \mathbf{M} &= \frac{1}{\lambda'} \boldsymbol{\alpha}_m \cdot \mathbf{H}^0, \quad \boldsymbol{\alpha}_m = V \boldsymbol{\chi}_m \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi}_m)^{-1}, \\
 \mathbf{H}^{\text{ext}}(\vec{r}) &= \mathbf{H}^0 - \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{M} \quad \text{for } \vec{r} \notin V.
 \end{aligned} \tag{2.49}$$

Conductors:

$$\begin{aligned}
 \mathbf{E}^{\text{dep}} &= -\mathbf{N} \cdot \mathbf{E}^+ \\
 \mathbf{E}^{\text{int}} &= \mathbf{E}^0 - \mathbf{N} \cdot \mathbf{E}^+ = 0, \quad \mathbf{E}^+ = \mathbf{N}^{-1} \cdot \mathbf{E}^0 \quad \text{for } \vec{r} \in V, \\
 \mathbf{p} &= \frac{\varepsilon_0}{\lambda} \boldsymbol{\alpha}_c \cdot \mathbf{E}^0, \quad \boldsymbol{\alpha}_c = V \mathbf{N}^{-1}, \\
 \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{E}^+ \quad \text{for } \vec{r} \notin V.
 \end{aligned} \tag{2.63}$$

Superconductors (magnetization model):

$$\begin{aligned}
 \mathbf{B}^{\text{int}} &= \mu_0 (\mathbf{H}^{\text{int}} + \lambda' \mathbf{M}) = 0, \quad \mathbf{M} = -\frac{1}{\mu_0 \lambda'} (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{B}^0 \quad \text{for } \vec{r} \in V, \\
 \mathbf{m} = V \cdot \mathbf{M} &= \frac{1}{\mu_0 \lambda'} \boldsymbol{\alpha}_s \cdot \mathbf{B}^0, \quad \boldsymbol{\alpha}_s = -V (\mathbf{1} - \mathbf{N})^{-1}, \\
 \mathbf{B}^{\text{ext}}(\vec{r}) &= \mathbf{B}^0 - \mu_0 \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{M} \quad \text{for } \vec{r} \notin V.
 \end{aligned} \tag{2.73}$$

Superconductors (surface current model):

$$\begin{aligned}
 \mathbf{B}^{\text{dep}} &= -(\mathbf{1} - \mathbf{N}) \cdot \mathbf{B}^+ \\
 \mathbf{B}^{\text{int}} = \mathbf{H}^{\text{int}} = 0 = \mathbf{M}, \quad \mathbf{B}^+ &= (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{B}^0, \\
 \mathbf{m} &= \frac{1}{\mu_0 \lambda_s} \boldsymbol{\alpha}_s \cdot \mathbf{B}^0, \quad \boldsymbol{\alpha}_s = -V(\mathbf{1} - \mathbf{N})^{-1}, \\
 \mathbf{B}^{\text{ext}}(\vec{r}) &= \mathbf{B}^0 + \mathbf{n}^{\text{ext}}(\vec{r}) \cdot (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{B}^0.
 \end{aligned} \tag{2.84}$$

General formulation of the method

Along the sections of this chapter a study was made of the electromagnetic behaviour of solid and homogeneous ellipsoidal bodies —either isotropic or anisotropic— under an applied static and uniform electric or magnetic field. In what follows a general formulation of the treatment is given in matrix representation, showing that the ensuing integro-differential equations are essentially identical in all cases and may all be solved, for the linear case, in exactly the same way. The main discussion is devoted to induced polarizations because permanent ones require specific analysis.

In all cases a vectorial field \mathbf{F} —electric field \vec{E} , magnetic field \vec{H} , or magnetic induction \vec{B} — is derived from a static applied one \mathbf{F}^0 through a vectorial magnitude $\mathbf{Q}(\mathbf{F})$ —electric polarization, magnetization or step discontinuity through the body's surface— which is a linear function of \mathbf{F} . $\mathbf{Q}(\mathbf{F})$ fully describes the state of uniform polarization of the body, which may be alternately characterized by some surface density of charge or current. The linear relationship between \mathbf{Q} and \mathbf{F} is either described by a susceptibility —case of the linear but perhaps anisotropic relationship $\mathbf{Q} = \boldsymbol{\chi} \cdot \mathbf{F}$ — or by a condition of cancellation of the internal field. Volume integrals are involved in most cases, but conductors and superconductors initially require surface integrals that may be transformed into volume ones by defining appropriate equivalent polarizations. The ensuing integro-differential equations, valid over all space, are separately solved for the region occupied by the body and the external one. It is then obtained

$$\begin{aligned}
 \mathbf{F}^{\text{int}} &= \mathbf{F}^0 - \mathbf{N} \cdot \mathbf{Q}(\mathbf{F}^{\text{int}}) \quad \text{for } \vec{r} \in V, \\
 \mathbf{F}^{\text{ext}}(\vec{r}) &= \mathbf{F}^0 - \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{Q}(\mathbf{F}^{\text{int}}) \quad \text{for } \vec{r} \notin V.
 \end{aligned} \tag{2.85}$$

The depolarization tensor \mathbf{n} , eq. 3.4, is uniform inside an ellipsoidal body (where its position independent value is \mathbf{N}). \mathbf{F}^{int} is therefore also uniform there, its value being determined by the first equation, as well as that of $\mathbf{Q}(\mathbf{F}^{\text{int}})$. The value of the external field then follows from the second of eqs. 2.85. The geometric contribution is determined by \mathbf{N} and \mathbf{n}^{ext} , while the material's properties are characterized by \mathbf{Q} .

The explicit values of \mathbf{F}^{int} and \mathbf{Q} are obtained in different ways according to the material's properties. The first one is the use of the linear relationship

$$\mathbf{Q}(\mathbf{F}^{\text{int}}) = \chi \cdot \mathbf{F}^{\text{int}}, \quad (2.86)$$

where χ is a susceptibility. In this case

$$\boxed{\mathbf{F}^{\text{int}} = (\mathbf{1} + \mathbf{N} \cdot \chi)^{-1} \cdot \mathbf{F}^0, \quad \mathbf{Q} = \chi \cdot \mathbf{F}^{\text{int}} = \chi \cdot (\mathbf{1} + \mathbf{N} \cdot \chi)^{-1} \cdot \mathbf{F}^0.} \quad (2.87)$$

The second case applies when the internal field vanishes (see Table 1):

$$\mathbf{F}^0 - \mathbf{N} \cdot \mathbf{Q} = 0 \quad \text{or} \quad (\mathbf{1} - \mathbf{N}) \cdot \mathbf{Q} = \mathbf{F}^0. \quad (2.88)$$

These relationships correspond to conductors and superconductors, where the susceptibility obtains its largest value

$$\mathbf{Q} = -\mathbf{N}^{-1} \cdot \mathbf{F}^0 \quad \text{or} \quad \mathbf{Q} = (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{F}^0. \quad (2.89)$$

The surface conduction current model of superconductors is the only case for which no single \mathbf{Q} tensor may be defined for both the internal and external field. The origin of this asymmetry is the difference between the constitutive equations for the electric and the magnetic case, eqs. 2.11 and 2.35.

In the following table magnitudes \mathbf{F} and \mathbf{Q} are identified for all cases discussed, as well as the linear functions and cancellations required for the validity of the treatment. Although they do not satisfy eqs. 2.87 to 2.89, permanent polarizations have been included in order to give a full panoramic view of the method.

Material	\mathbf{F}	\mathbf{Q}	χ	Condition	Eq.
Ferroelectric	\mathbf{E}	$\frac{\lambda}{\epsilon_0} \mathbf{P}$		$\mathbf{P} = \mathbf{P}(\mathbf{E}^{\text{int}})$	2.12
Dielectric	\mathbf{E}	$\frac{\lambda}{\epsilon_0} \mathbf{P}$	χ_e	$\mathbf{P} = \frac{\epsilon_0}{\lambda} \chi_e \cdot \mathbf{E}^{\text{int}}$	2.33
Conductor (surface charge)	\mathbf{E}	\mathbf{E}^+		$\mathbf{E}^+ = \mathbf{N}^{-1} \cdot \mathbf{E}^0$	2.63
Conductor (equivalent \mathbf{P})	\mathbf{E}	$4\pi k_1 \mathbf{P}$	$\chi = \infty$	$\mathbf{P} = \chi_e \mathbf{E}^{\text{int}}$	2.65
Ferromagnetic	\mathbf{H}	$\lambda' \mathbf{M}$		$\mathbf{M} = \mathbf{M}(\mathbf{H}^{\text{int}})$	2.44
Diamagnetic and paramagnetic	\mathbf{H}	$\lambda' \mathbf{M}$	χ_m	$\mathbf{M} = \chi_m \cdot \mathbf{H}^{\text{int}}$	2.49
Superconductor (equivalent \mathbf{M})	\mathbf{H}	$\lambda' \mathbf{M}$	$\chi_m = -1$	$\mathbf{M} = \chi_m \mathbf{H}^{\text{int}}$	2.73
Superconductor (surface conduction current)	\mathbf{B}	$\mathbf{Q}^{\text{int}} = (\mathbf{N}^{-1} - \mathbf{1}) \cdot \mathbf{B}^+$ $\mathbf{Q}^{\text{ext}} = -\mathbf{B}^+$		$\mathbf{B}^{\text{int}} = 0$	2.84

Table 2. Identification of the magnitudes in the general field equations for different kinds of materials.

Solving the integro-differential equations by iteration

The dielectric case, eq. 2.31, may be taken as a prototype of the integro-differential equations which determine the fields:

$$\vec{E}(\vec{r}) = \vec{E}^0 + O(\vec{E}), \quad (2.90)$$

where O is a linear operator with the following properties (see eq. 3.3):

$$O(\vec{F}(\vec{r})) = k_1 \nabla \left(\nabla \cdot \iint_V \frac{\chi_e \cdot \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 r' \right) \quad (2.92)$$

$$\begin{aligned} O(k \cdot \vec{F}(\vec{r})) &= k \cdot O(\vec{F}(\vec{r})), \\ O(\vec{F}(\vec{r}) + \vec{G}(\vec{r})) &= O(\vec{F}(\vec{r})) + O(\vec{G}(\vec{r})), \\ O(\vec{F}^0) &= -4\pi \mathbf{N} \cdot \chi_e \cdot \vec{F}^0 \quad \text{if } \vec{r} \in V. \end{aligned} \quad (2.92)$$

The second member of the last equation holds true only when \vec{F}^0 is a uniform vector throughout the body's volume V .

The solution for each of these equations was previously found by using a uniform Ansatz, as discussed in the different sections. The resulting field coincides, in all known cases, with that obtained by solving Laplace's equation in ellipsoidal coordinates by the method of separation of variables⁸². The examples are only illustrative and one should prove that eq. 2.90 has a unique solution, as sketched below.

Let $\vec{F}^{(1)}$ and $\vec{F}^{(2)}$ be two different solutions of eq. 2.90, that is

$$\vec{F}^{(1)} = \vec{F}^0 + O(\vec{F}^{(1)}), \quad \vec{F}^{(2)} = \vec{F}^0 + O(\vec{F}^{(2)}), \quad \text{that is } \vec{F}^{(1)} - \vec{F}^{(2)} = \vec{G} = O(\vec{G}). \quad (2.93)$$

By iteration it is easily proved that

$$\vec{G} = O^n(\vec{G}) \quad \text{for } n = 1, 2, 3, \dots, \quad (2.94)$$

equation that is satisfied only for $\vec{G} = 0$, proving that the solution is unique.

The inhomogeneous eq. 2.90 may be solved by iteration, succesively replacing the argument of the operator by the full second member. The first replacement gives

$$\begin{aligned} \vec{E} &= \vec{E}^0 + O(\vec{E}^0 + O(\vec{E})) \\ &= O^0(\vec{E}^0) + O^1(\vec{E}^0) + O^2(\vec{E}) = \sum_{\alpha=0}^1 O^\alpha(\vec{E}^0) + O^2(\vec{E}), \end{aligned} \quad (2.95)$$

⁸² Full references to solutions are given in the section [History](#) of Chapter 1.

where O^n is the n th power of O .

The second iteration gives

$$\begin{aligned}\vec{E} &= \sum_{\alpha=0}^1 O^\alpha (\vec{E}^0) + O^2 \left(\sum_{\alpha=0}^1 O^\alpha (\vec{E}^0) + O^2 (\vec{E}) \right) \\ &= \sum_{\alpha=0}^1 O^\alpha (\vec{E}^0) + \sum_{\alpha=0}^1 O^{\alpha+2} (\vec{E}^0) + O^4 (\vec{E}) = \sum_{\alpha=0}^3 O^\alpha (\vec{E}^0) + O^4 (\vec{E}).\end{aligned}\quad (2.96)$$

In the infinite iterations limit one gets

$$\begin{aligned}\vec{E} &= \sum_{\alpha=0}^{\infty} O^\alpha (\vec{E}^0) + O^\infty (\vec{E}), \\ O^\alpha (\vec{E}^0) &= (-4k_1 \mathbf{N} \cdot \boldsymbol{\chi}_e)^\alpha \cdot \vec{E}^0,\end{aligned}\quad (2.97)$$

where the operator series coincides with the Taylor's expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots = \sum_{\alpha=0}^{\infty} (-x)^\alpha. \quad (2.98)$$

If the series converges and the last term of eq. 2.97 tends to 0, it is obtained

$$\vec{E}(\vec{r}) = (1 + \mathbf{N} \cdot \boldsymbol{\chi})^{-1} \cdot \vec{E}^0. \quad (2.99)$$

The expression concides with eqs. 2.33 and 2.87, showing that the iterative expansion converges and coincides with that obtained using a uniform Ansatz. Although theoretical physicists frequently use series expansions of operators, the method is seldom used by engineers. It is easier to give a rigorous justification of this kind of expansion by using matrices⁸³.

The same method may be used to solve the other three integro-differential equations in Table 1. This iterative method, clumsier than the use of a uniform Ansatz, is presented here only to stress the mathematical consistency of the use of integro-differential equations. It might, perhaps, be used as an approximate method for studying the case of non ellipsoidal bodies or equations 2.85 when \mathbf{Q} is not a linear function of \mathbf{F} . The exploration of this subject, not made by the the author, is left to the interested reader.

⁸³ See, for instance: G. Goertzel and N. Tralli; *Some Mathematical Methods of Physics*; McGraw-Hill; 1960; chapters 1, 2 and 3, pp. 7-49.

Chapter 3: The depolarization tensor: basic treatment

Definition

The depolarization tensor $\mathbf{n}(\vec{r})$ arises from expressions like

$$\begin{aligned} & \nabla(\nabla I(\vec{r}) \cdot \vec{F}), (\vec{F} \cdot \nabla)(\nabla I(\vec{r})), \nabla(\nabla \cdot I(\vec{r}) \vec{F}) \text{ and } \nabla \oint_S \frac{\vec{F} \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}, \\ & \text{where } I(\vec{r}) = \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} \text{ and } \vec{F} \text{ is uniform in } V. \end{aligned} \quad (3.1)$$

The integration is made over the volume V of an ellipsoidal body or its closed surface S , and \vec{r} is the position of a field point that may be either inside or outside the body. \vec{F} is a uniform vector inside the region of integration —the body's volume, including its interior surface— that may be an electric or magnetic field or a polarization. [Table 1](#) gives the list of the integro-differential equations discussed in this book where such expressions arise.

Only the first two expressions will be analyzed next as the third is very similar. The case where \mathbf{n} is derived from a surface integral is discussed at [section *n as a surface integral*](#). The proof is easier to follow if one writes \vec{F} and ∇ in terms of components and unit vectors, as follows:

$$\nabla = \sum_{\beta} \hat{x}_{\beta} \frac{\partial}{\partial x_{\beta}}, \quad \vec{F} = \sum_{\gamma} \hat{x}_{\gamma} F_{\gamma}, \quad \text{where } \hat{x}_{\beta} \cdot \hat{x}_{\gamma} = \delta_{\beta\gamma}. \quad (3.2)$$

Then

$$\begin{aligned} \nabla(\nabla I(\vec{r}) \cdot \vec{F}) &= \sum_{\alpha} \hat{x}_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left(\sum_{\beta} \hat{x}_{\beta} \frac{\partial I(\vec{r})}{\partial x_{\beta}} \cdot \sum_{\gamma} \hat{x}_{\gamma} F_{\gamma} \right) \\ &= \sum_{\alpha} \hat{x}_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left(\sum_{\beta, \gamma} (\hat{x}_{\beta} \cdot \hat{x}_{\gamma}) \frac{\partial I(\vec{r})}{\partial x_{\beta}} F_{\gamma} \right) = \sum_{\alpha} \hat{x}_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left(\sum_{\beta, \gamma} \delta_{\beta\gamma} \frac{\partial I(\vec{r})}{\partial x_{\beta}} F_{\gamma} \right) \\ &= \sum_{\alpha} \hat{x}_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left(\sum_{\beta} \frac{\partial I(\vec{r})}{\partial x_{\beta}} F_{\beta} \right) = \sum_{\alpha, \beta} \hat{x}_{\alpha} \frac{\partial^2 I(\vec{r})}{\partial x_{\alpha} \partial x_{\beta}} F_{\beta}. \end{aligned} \quad (3.3)$$

One may now define the components of the rank two depolarization tensor

$$n_{\alpha\beta}(\vec{r}) = -\frac{1}{4\pi} \frac{\partial^2 I(\vec{r})}{\partial x_\alpha \partial x_\beta} = -\frac{1}{4\pi} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|}, \quad (3.4)$$

that in dyadic representation (see [Appendix 6](#)) may be written

$$\mathbf{n}(\vec{r}) = \sum_{\alpha,\beta} \hat{x}_\alpha n_{\alpha\beta}(\vec{r}) \hat{x}_\beta. \quad (3.5)$$

For instance

$$\begin{aligned} (\vec{F} \cdot \nabla) \nabla I(\vec{r}) &= \left(\sum_\alpha F_\alpha \frac{\partial}{\partial x_\alpha} \right) \sum_\beta \frac{\partial I(\vec{r})}{\partial x_\beta} \hat{x}_\beta = \sum_{\alpha,\beta} F_\alpha \frac{\partial^2 I(\vec{r})}{\partial x_\alpha \partial x_\beta} \hat{x}_\beta \\ &= -4\pi \vec{F} \cdot \sum_{\alpha,\beta} \hat{x}_\alpha n_{\alpha\beta}(\vec{r}) \hat{x}_\beta = -4\pi \mathbf{n}(\vec{r}) \cdot \vec{F}, \end{aligned} \quad (3.6)$$

as the depolarization tensor is symmetric (see eq. [3.12](#)).

Therefore,

$$\begin{aligned} (\vec{F} \cdot \nabla) \nabla I(\vec{r}) &= \nabla (\nabla \cdot I(\vec{r}) \vec{F}) = \nabla (\nabla I(\vec{r}) \cdot \vec{F}) = -\nabla \iiint_s \frac{\vec{F} \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} \\ &= -4\pi \mathbf{n}(\vec{r}) \cdot \vec{F}, \text{ where } \mathbf{n}(\vec{r}) = \sum_{\alpha,\beta} \hat{x}_\alpha n_{\alpha\beta} \hat{x}_\beta. \end{aligned} \quad (3.7)$$

The calculation of the depolarization tensor \mathbf{n} is therefore reduced to the calculation of the integral $I(\vec{r})$ eq. [3.3](#) for different kinds of ellipsoidal bodies. For a solid ellipsoidal body this integral is, apart from a constant factor, either the potential for a uniformly charged one, or the gravitational potential for a constant mass density. In this chapter the first potential is used, as it is more familiar to students of engineering, and the second potential —the less familiar one with more complex derivation— is used in the following chapter to derive expressions for \mathbf{n} .

It is convenient to define an auxiliary function $f(\vec{r})$ such that eq. [3.4](#) reduces to

$$n_{\alpha\beta}(\vec{r}) = \frac{\partial^2 f(\vec{r})}{\partial x_\alpha \partial x_\beta}, \text{ where } f(\vec{r}) = -\frac{1}{4\pi} \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} = -\frac{\phi(\vec{r})}{4\pi k\sigma}. \quad (3.8)$$

In the last term ϕ is a gravitational or electrostatic potential, σ the constant mass or charge density and k the gravitational or electrostatic constant determined by the choice of system of units. That is, they are the factors such that

$$\frac{1}{k\sigma}\phi(\vec{r}) = \iiint_V \frac{d^3r'}{|\vec{r} - \vec{r}'|} \quad (3.9)$$

Tensor $\mathbf{n}(\vec{r})$ is uniform inside ellipsoidal bodies, where it was originally defined⁸⁴. The author of this book was apparently the first one to define it outside these bodies although restricted to the magnetic case⁸⁵. In the literature of magnetism its internal value is sometimes called demagnetization tensor (apart from a possible denominator 4π), its principal values being the well known demagnetizing or demagnetization coefficients or factors⁸⁶. The tensor is indistinctly written here as a dyadic \mathbf{n} or a 3x3 matrix \mathbf{n} .

For applications it is necessary to explicitly identify the regions inside and outside the ellipsoidal body. To that end \mathbf{N} (dyadic) or \mathbf{N} (matrix) denote the uniform internal depolarization tensor and $\mathbf{n}^{\text{ext}}(\vec{r})$ or $\mathbf{n}^{\text{ext}}(\vec{r})$ the non-uniform external one

$$\mathbf{N} = \sum_{\alpha} \sum_{\beta} \hat{x}_{\alpha} \hat{x}_{\beta} N_{\alpha\beta} \quad \text{for } \vec{r} \in V, \quad \mathbf{n}^{\text{ext}}(\vec{r}) = \sum_{\alpha} \sum_{\beta} \hat{x}_{\alpha} \hat{x}_{\beta} n_{\alpha\beta}(\vec{r}) \quad \text{for } \vec{r} \notin V. \quad (3.10)$$

$$\mathbf{N} = \begin{pmatrix} N_{xx} & N_{xy} & N_{xz} \\ N_{xy} & N_{yy} & N_{yz} \\ N_{xz} & N_{yz} & N_{zz} \end{pmatrix} \quad \text{for } \vec{r} \in V, \quad (3.11)$$

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \begin{pmatrix} n_{xx}(\vec{r}) & n_{xy}(\vec{r}) & n_{xz}(\vec{r}) \\ n_{xy}(\vec{r}) & n_{yy}(\vec{r}) & n_{yz}(\vec{r}) \\ n_{xz}(\vec{r}) & n_{yz}(\vec{r}) & n_{zz}(\vec{r}) \end{pmatrix} \quad \text{for } \vec{r} \notin V.$$

The difference between both regions is emphasized by explicitly writing the dependance on position for the external case and omitting it in the internal case, as for constants or parameters.

General properties of \mathbf{n}

Symmetric tensor

As the order of derivation may be freely exchanged⁸⁷, \mathbf{n} turns out to be a symmetric tensor:

$$n_{\alpha\beta}(\vec{r}) = n_{\beta\alpha}(\vec{r}), \quad (3.12)$$

⁸⁴ Landau and Lifchitz, p. 44. Maxwell, apparently the first to introduce it, did not identify its tensorial character.

⁸⁵ Soliv  rez (1981).

⁸⁶ Moskowitz and Della Torre, p. 739.

⁸⁷ MacMillan, pp. 27-32.

property that was previously used in eq. 3.6.

Trace

From eq. 3.4 the trace of \mathbf{n} (sum of its diagonal elements) is

$$\text{Tr } \mathbf{n}(\vec{r}) = \sum_{\alpha} n_{\alpha\alpha}(\vec{r}) = \sum_{\alpha} \frac{\partial^2 f(\vec{r})}{\partial x_{\alpha}^2} = \Delta f(\vec{r}) = -\frac{1}{4\pi} \nabla \cdot \nabla \iiint_V \frac{1}{|\vec{r} - \vec{r}'|} d^3r', \quad (3.13)$$

where Δ is the Laplacian operator eq. A2.3. The value of this operation, frequently used in the theory of electromagnetism, is an integrable singularity inside V and vanishes outside the body. As the calculation is often made in a questionable way, in Appendix 3 (see eq. A3.7) it is proved that⁸⁸.

$$\Delta \iiint_V \frac{d^3r'}{|\vec{r} - \vec{r}'|} = \begin{cases} -4\pi & \text{for } \vec{r} \in V \\ 0 & \text{for } \vec{r} \notin V \end{cases}. \quad (3.14)$$

It follows that

$$\boxed{\text{Tr } \mathbf{N} = 1, \quad \text{Tr } \mathbf{n}^{\text{ext}} = 0.} \quad (3.15)$$

In fact, the constants in eq. 3.8 were chosen in order to make the trace of \mathbf{N} to be 1.

Orthogonal transformations

The transformation properties of matrix \mathbf{n} are now analyzed under an orthogonal change (rotations, reflections, inversions and all its combinations) of coordinate system:

$$x'_{\alpha} = \sum_{\beta} R_{\alpha\beta} x_{\beta}, \quad \mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{R} \cdot \mathbf{r}. \quad (3.16)$$

These transformations are the basis of the use of symmetry properties in order to simplify the calculation of many physical properties (see the section [Symmetries](#), and [Problem 19](#)).

Such operations are always represented by an orthogonal matrix \mathbf{R} , that is, one such that its inverse coincides with its transposed matrix:

$$\mathbf{R}^{-1} = \mathbf{R}^t, \quad \mathbf{R} \cdot \mathbf{R}^t = \mathbf{R}^t \cdot \mathbf{R} = \mathbf{1}, \quad \text{where } R_{\alpha\beta}^t = R_{\beta\alpha}. \quad (3.17)$$

⁸⁸ V. Hnizdo, Eur. J. Phys. vol. **21**, pp. L1-L3 (2000) gives a rigorous but more complex proof.

The inverse transformation is then

$$\mathbf{r} = \mathbf{R}^{-1} \cdot \mathbf{r}' = \mathbf{R}^t \cdot \mathbf{r}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{yx} & R_{zx} \\ R_{xy} & R_{yy} & R_{zy} \\ R_{xz} & R_{yz} & R_{zz} \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad x_\alpha = \sum_\beta R_{\beta\alpha} x'_\beta. \quad (3.18)$$

In order to determine the components of \mathbf{n} in the new coordinate system, one should use definition eq. 3.8,

$$n_{\alpha\beta}(\vec{r}) = \frac{\partial^2 f(\vec{r})}{\partial x_\alpha \partial x_\beta}, \quad (3.19)$$

and express the value of the new partial derivatives in terms of the original ones⁸⁹. In terms of the derivative operator:

$$\frac{\partial}{\partial x'_\beta} = \sum_\alpha \frac{\partial}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial x'_\beta} = \sum_\alpha R_{\beta\alpha} \frac{\partial}{\partial x_\alpha}. \quad (3.20)$$

It is thus obtained for the new matrix \mathbf{N}'

$$N'_{\alpha\beta} = \frac{\partial^2 f}{\partial x'_\alpha \partial x'_\beta} = \sum_{\gamma,\delta} R_{\alpha\gamma} \frac{\partial^2 f}{\partial x_\gamma \partial x_\delta} R_{\beta\delta} = (\mathbf{R} \cdot \mathbf{N} \cdot \mathbf{R}^t)_{\alpha\beta} \quad \text{or} \quad \mathbf{N}' = \mathbf{R} \cdot \mathbf{N} \cdot \mathbf{R}^t. \quad (3.21)$$

where eq. 3.17 was used.

By using this transformation property one may establish relationships among components of the polarization tensor, topic discussed in the following section.

If one determines the eigenvalues of \mathbf{n} in its principal coordinate system, the value of \mathbf{n}' in a coordinate system related to the former by eq. 3.21 is

$$\begin{aligned} \mathbf{n}' &= \mathbf{R} \cdot \mathbf{n} \cdot \mathbf{R}^t \\ &= \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \cdot \begin{pmatrix} n_x(\vec{r}) & 0 & 0 \\ 0 & n_y(\vec{r}) & 0 \\ 0 & 0 & n_z(\vec{r}) \end{pmatrix} \cdot \begin{pmatrix} R_{xx} & R_{yx} & R_{zx} \\ R_{xy} & R_{yy} & R_{zy} \\ R_{xz} & R_{yz} & R_{zz} \end{pmatrix}. \end{aligned} \quad (3.22)$$

Symmetries

The relationships among the constant eigenvalues of \mathbf{N} determined by the body's symmetries—quite useful for applications—can be found by using the transformation properties eq. 3.22. The symmetries of the external depolarization tensor \mathbf{n}^{ext} are more difficult to treat because its components usually are cumbersome functions of position. This case is usually handled by using

⁸⁹ Santaló, pp. 299-303.

appropriate curvilinear coordinate systems (see, for instance, the treatment of the sphere in page 58).

When an orthogonal transformation \mathbf{R} is a symmetry of the ellipsoidal body, the value \mathbf{N}' in the transformed coordinate system must be equal to the original one:

$$\mathbf{N}' = \mathbf{R} \bullet \mathbf{N} \bullet \mathbf{R}^t = \mathbf{N}, \text{ or } \mathbf{R} \bullet \mathbf{N} = \mathbf{N} \bullet \mathbf{R}, \quad (3.23)$$

From a practical point of view it is simpler to use the conmutation relationship than the first one, as is illustrated next.

The case of spheroids (ellipsoids of revolution) is discussed now as an example. If the axis of revolution is taken as coordinate axis z , the exchange of axes x and y (a reflection plane) should leave \mathbf{N} unchanged. Therefore, for

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R} \bullet \mathbf{N} = \begin{pmatrix} 0 & N_y & 0 \\ N_x & 0 & 0 \\ 0 & 0 & N_z \end{pmatrix} = \mathbf{N} \bullet \mathbf{R} = \begin{pmatrix} 0 & N_x & 0 \\ N_y & 0 & 0 \\ 0 & 0 & N_z \end{pmatrix}. \quad (3.24)$$

It thus turns out that $N_x = N_y$ and no further relationships are found using other symmetry operations. The diagonal expression of \mathbf{N} for spheroids is thus

$$\mathbf{N}_{\text{spheroid}} = \begin{pmatrix} N_e & 0 & 0 \\ 0 & N_e & 0 \\ 0 & 0 & N_p \end{pmatrix} \quad (3.25)$$

N_p being the polar eigenvalue corresponding to the spheroid's symmetry axis, and N_e that of the two equal equatorial ones.

In the case of the sphere an exchange of axis y and z is also a symmetry, which leads to the three eigenvalues being equal, as discussed in Problem 19. In combination with the trace rule eqs. 3.15, one then obtains

$$N_x = N_y = N_z = N, \text{ Tr } \mathbf{N} = 3N = 1 \text{ and } N = 1/3. \quad (3.26)$$

The sphere is the only type of ellipsoid where the actual eigenvalues of \mathbf{N} may be determined by using only the trace rule and symmetry arguments. There are only two other cases where this can be done without finding primitives of integrals, but both require the knowledge of the integral expressions of N_ω , topic discussed in some sections of this chapter and the next.

***n* as a surface integral**

The depolarization tensor \mathbf{n} may be expressed as a surface integral in cases involving surface densities of charge or current. To that end one should take the scalar product of eq. 3.4 with a uniform vector \vec{C} :

$$\begin{aligned}
\mathbf{n}(\vec{r}) \cdot \vec{C} &= \sum_{\alpha, \beta} \hat{x}_{\alpha} n_{\alpha\beta}(\vec{r}) \hat{x}_{\beta} \cdot \vec{C} = \sum_{\alpha, \beta} \hat{x}_{\alpha} n_{\alpha\beta}(\vec{r}) C_{\beta} \\
&= -\frac{1}{4\pi} \sum_{\alpha, \beta} C_{\beta} \hat{x}_{\alpha} \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} = \frac{1}{4\pi} \nabla \left(\left(\iiint_V \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 r' \right) \cdot \vec{C} \right).
\end{aligned} \tag{3.27}$$

As $I(\vec{r})$ (eq. 3.8) is a continuous function, the gradient theorem eq. A3.9 may be used to obtain

$$\iiint_V \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3 r' = \oint_S \frac{d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}. \tag{3.28}$$

As \vec{C} is an arbitrary vector, it follows that

$$\boxed{\mathbf{n}(\vec{r}) = \frac{1}{4\pi} \nabla \oint_S \frac{d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}}, \tag{3.29}$$

an expression previously given by Moskowitz and Della Torre⁹⁰.

Surface step discontinuity

The depolarization tensor has a step discontinuity through the body's surface, whose characteristics are better understood by analyzing a concrete case, here taken to be the electric one. As discussed next, it turns out that in this case the discontinuity is equivalent to a charge density. In the magnetic case —where the analysis is quite similar— it corresponds to a surface current density. In [Problem 30](#) the reader is challenged to find the expression for the surface discontinuity without using the properties of electric or magnetic fields.

Surface charge density

For a uniform electric polarization \vec{P} —whether it is spontaneous, induced or equivalent— the resulting electric field may always be written as (see eq. 2.9)

$$\vec{E}_p(\vec{r}) = -\nabla \phi_p(\vec{r}) \quad \text{where} \quad \phi_p(\vec{r}) = k_1 \left(\iiint_V \nabla' \frac{d^3 r'}{|\vec{r} - \vec{r}'|} \right) \cdot \vec{P}. \tag{3.30}$$

As $I(\vec{r})$ (eq. 3.8) is a continuous function, the gradient theorem eq. A3.9 may be used to reduce the volume integral to the following surface integral over S :

⁹⁰ Moskowitz and Della Torre, p. 740 eq. 17.

$$\begin{aligned}
\phi_p(\vec{r}) &= k_1 \left(\iiint_V \nabla' \cdot \frac{d^3 \vec{r}'}{|\vec{r} - \vec{r}'|} \right) \cdot \vec{P} = k_1 \iiint_S \frac{d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} \cdot \vec{P} \\
&= k_1 \iiint_S \frac{\sigma_p(\vec{r}') d^2 \vec{r}'}{|\vec{r} - \vec{r}'|}, \text{ where } \sigma_p(\vec{r}') = \hat{s}(\vec{r}') \cdot \vec{P}.
\end{aligned} \tag{3.31}$$

The last integral expresses the potential created by a surface charge density σ_p^{91} .

Using eq. A7.8 it is obtained

$$\sigma_p(\vec{r}) = \hat{s}(\vec{r}) \cdot \vec{P} = \frac{\frac{xP_x}{a^2} + \frac{yP_y}{b^2} + \frac{zP_z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}, \tag{3.32}$$

which is the electric charge density created by the step discontinuity on the body's surface, including the equivalent polarization case in the case of conductors. This expression, useful for ellipsoidal conductors, is not given in any of the textbooks used for writing of this book⁹². The validity of eq. 3.32 may be easily verified for the special case of the sphere discussed in Problem 09.

Surface step discontinuity

Through a material interphase with a surface charge density σ of net (case not considered in this book) or induced charge the component of an electric field parallel to the surface is continuous, but the normal component is not. If $\hat{s}(\vec{r}_s)$ is the unit vector normal to the surface at point \vec{r}_s , the decomposition of any vector in components parallel and normal to that surface is:

$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}, \quad \vec{F}_{\parallel} = (\hat{s} \cdot \vec{F}) \hat{s} = \hat{s} \hat{s} \cdot \vec{F}, \quad \vec{F}_{\perp} = (\mathbf{1} - \hat{s} \hat{s}) \cdot \vec{F}, \tag{3.33}$$

where $\mathbf{1}$ is the unit dyadic of eq. A6.4.

The continuity of the electric field component tangential to the body's surface is then expressed in the following way⁹³:

$$(\mathbf{1} - \hat{s} \hat{s}) \cdot (\vec{E}^+ - \vec{E}^-) = (\vec{E}^+ - \vec{E}^-) \cdot (\mathbf{1} - \hat{s} \hat{s}) = 0. \tag{3.34}$$

The discontinuity of the component normal to the body's surface S is⁹⁴:

⁹¹ Reitz, p. 31 eq. 2-15.

⁹² Stratton —the autor that discusses in more detail the electromagnetic properties of ellipsoidal bodies—gives only the surface density for charged conductors in the absence of an applied field, see eq. 12 in p. 209.

⁹³ Reitz, p. 90.

⁹⁴ Stratton, p. 188.

$$(\vec{E}^+ - \vec{E}^-) \cdot \hat{s} = 4\pi k_1 \sigma. \quad (3.35)$$

The signs are chosen so that a component is positive when the vector points outgoing from the positive side of the surface (+), chosen to be the one external to the body.

For all the cases discussed in the previous chapter —see eqs. 2.7 and A1.4— the following relationship is fulfilled,

$$\vec{E}^+ - \vec{E}^- = 4\pi k_1 (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) \cdot \vec{P}, \quad (3.36)$$

where \vec{r}^S is a point on surface S .

From this identity and the expression of \hat{s} eq. 3.31, the field's discontinuity through the body's surface, the depolarization tensor's step discontinuity, may be written as

$$\hat{s} \cdot (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) \cdot \vec{P} = \hat{s} \cdot \vec{P}. \quad (3.37)$$

For the equation to be valid for arbitrary values of \vec{P} —whether it is a spontaneous, induced or equivalent polarization— it should be

$$\hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S) \cdot (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) = (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) \cdot \hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S). \quad (3.38)$$

Any tensor \mathbf{T} may be decomposed in the following fashion

$$\mathbf{T} = (\mathbf{1} + \hat{s}\hat{s} - \hat{s}\hat{s}) \cdot \mathbf{T} = \hat{s}\hat{s} \cdot \mathbf{T} + (\mathbf{1} - \hat{s}\hat{s}) \mathbf{T} = \mathbf{T}_{\parallel} + \mathbf{T}_{\perp}, \quad (3.39)$$

in components such that when the scalar product with an arbitrary vector is taken, the components parallel and normal to the surface are respectively given by each term.

From eqs. 3.34 and 3.35 one gets

$$\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S) = \hat{s}\hat{s} \cdot (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) + (\mathbf{1} - \hat{s}\hat{s}) \cdot (\mathbf{N} - \mathbf{n}^{\text{ext}}(\vec{r}^S)) = \hat{s}\hat{s}, \quad (3.40)$$

which gives the value on the body's surface of the external depolarization. The following matrix expression is thus obtained, valid for all types of ellipsoids:

$$\mathbf{n}^{\text{ext}}(\vec{r}^S) = \mathbf{N} - \hat{s}(\vec{r}^S) \cdot \hat{s}(\vec{r}^S). \quad (3.41)$$

From eq. A7.8,

$$s_{\alpha}(\vec{r}^S) s_{\beta}(\vec{r}^S) = \frac{x_{\alpha}^S x_{\beta}^S}{\frac{(x^S)^2}{a^4} + \frac{(y^S)^2}{b^4} + \frac{(z^S)^2}{c^4}}. \quad (3.42)$$

At an elementary level the expression may be verified for all ellipsoids where the components of the external depolarization tensor may be expressed in terms of

elementary functions, like the infinite cylinder and the sphere (see [Problem 34](#)). Later on, it will be verified for all types of ellipsoids, including the triaxial one (see eq. 4.66). The formula provides a simple approximation to the fields near the surface of any ellipsoid. This includes the electric fields at sharp metallic tips, a topic where explicit formulas are rarely given (see [Problem 12](#)).

Calculation of \mathbf{n} using electrostatic Gauss's Law

This section is devoted to the calculation of \mathbf{n} from eq. 3.8, which requires the expression of the auxiliary integral $f(\vec{r})$. As stated there, apart from a proportionality constant f coincides with the electric potential ϕ generated by volume V when charged with uniform density ρ ⁹⁵:

$$\phi(\vec{r}) = k_1 \rho \iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} = -4\pi k_1 \rho f(\vec{r}), \quad f(\vec{r}) = -\frac{\phi(\vec{r})}{4\pi k_1 \rho}. \quad (3.43)$$

This expression is useful for the shapes where the potential may be derived by using electrostatic Gauss' law, high symmetry cases which are frequently discussed in first university courses on electricity and magnetism for physicist and engineers. In order of increasing difficulty these shapes are:

- $a = b \gg c$: constant thickness sheet of infinite extension;
- $a \gg b = c$: right circular cylinder of infinite length;
- $a = b = c$: sphere.

Sheet of constant thickness and infinite extension

The depolarization tensor of a sheet of constant thickness $e = 2c$ and infinite extension, as shown in the figure, is calculated here. A cartesian coordinate system is used such that its z axis is perpendicular to the surface of the sheet, with origin on its middle plane.

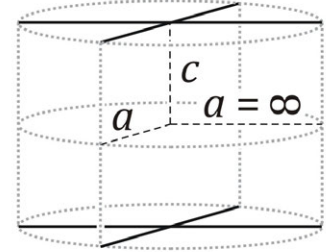


Figure 3. Infinite sheet.

The simplest way to solve the equivalent electrostatic problem of eq. 3.43 is to evaluate the electric field by application of electrostatic Gauss's Law, obtaining then the potential by integration. This may be easily done due to the high symmetry of the problem. As the sheet is invariant under arbitrary translations along the axes x, y , the field may only be a function of z . As it is also invariant under arbitrary rotations around the z axis, the field cannot have x and y components. The last symmetry of interest is the reflection plane parallel to the sheet's surfaces through its middle, making the field in one of its sides the specular image of that in the other. Therefore

$$\vec{E}(-z) = -\vec{E}(z), \quad \vec{E}(\vec{r}) = E(z)\hat{z}, \quad E(0) = 0. \quad (3.44)$$

⁹⁵ A similar expression applies for f in the gravitational case discussed by Macmillan and Kellog, where one should take $k_1 = 1$ and ρ to be the density of mass.

Figure 4 shows the sheet's cross section in gray and the two closed surfaces, S_1 and S_2 , used for the application of Gauss's Law. These gaussian boxes are right circular cylinders such that their vertical faces are parallel to axis z and the horizontal ones, of area A , are normal to it. The flux of the electric field vanishes both on the base (where $\vec{E} = 0$) and on the vertical face (where it is normal to the surface). This reduces the calculation to the flux on the upper face, where

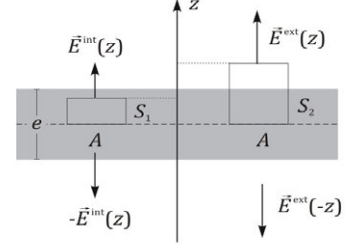


Figure 4. Gaussian boxes used for solving the infinite sheet.

$$\begin{aligned} \oint_{S_1} \vec{E}^{\text{int}} \cdot d\vec{S} &= 4\pi k_1 Q_1, \text{ gives } E^{\text{int}}(z) \cdot A = 4\pi k_1 \rho \cdot A \cdot z, \quad E^{\text{int}}(z) = 4\pi k_1 \rho \cdot z, \\ \oint_{S_2} \vec{E}^{\text{ext}} \cdot d\vec{S} &= 4\pi k_1 Q_2, \text{ gives } E^{\text{ext}}(z) \cdot A = 4\pi k_1 \rho \cdot A \cdot e, \quad E^{\text{ext}}(z) = 4\pi k_1 \rho \cdot e. \end{aligned} \quad (3.45)$$

From $E(z) = -\frac{d\phi(z)}{dz}$ and eq. 3.43 it is obtained

$$\phi(z) = \begin{cases} \text{cte.} - 2\pi k_1 \rho \cdot z^2 & \text{for } \vec{r} \in V \\ \text{cte.} - 2\pi k_1 \rho \cdot e \cdot z & \text{for } \vec{r} \notin V \end{cases}, \quad f(z) = \begin{cases} \frac{1}{2} z^2 & \text{for } \vec{r} \in V, \\ \frac{1}{2} e \cdot z & \text{for } \vec{r} \notin V \end{cases} \quad (3.46)$$

From the definition eq. 3.4 it follows that

$$\text{for the sheet} \quad \boxed{\mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{n}^{\text{ext}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.} \quad (3.47)$$

Due to the inexistence of \mathbf{N}^{-1} one should next analyze the relationship of the fields thus obtained with those of the depolarization tensor method. Equations 2.33 and 2.49 show that the relevant matrix for the case of induced electric and magnetic polarizations is

$$\begin{aligned} (\mathbf{1} + \chi \mathbf{N})^{-1} &= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi \end{pmatrix} \right]^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \chi \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(1 + \chi) \end{pmatrix}, \end{aligned} \quad (3.48)$$

where it is clear no problem arises. This does not happen for the case of conductors and superconductors, eqs. 2.63, 2.73 and 2.84. In these cases one

should go back to the original equation that was solved by the inversion of a matrix. For instance, the first of eqs. 2.63 leads to the equation

$$\mathbf{N} \cdot \mathbf{E}^+ = \mathbf{E}^0, \quad \text{that is} \quad E_x^+ = 0, E_y^+ = 0, E_z^+ = E_z^0, \quad (3.49)$$

where \mathbf{E}^0 is the applied electric field and \mathbf{E}^+ is the total electric field on the external surface of the sheet due to the surface density of charge⁹⁶ induced by \mathbf{E}^0 . The physical meaning of the previous equation is that, as regards the “polarization” effects —those that have origin in the redistribution of surface charges— the only component of the applied field to be taken into account is the one normal to the sheet, the z component. In the reduced space of the z axis, matrix \mathbf{N} has what may be called a “reduced” inverse (this is better seen in the two-dimensional case discussed at the end of next section).

Right circular cylinder of infinite length

As in the previous problem, the value of \mathbf{n} will be derived from the calculation of the electric field of a uniformly charged right circular cylinder of radius b , infinite length and uniform electric charge density ρ . By symmetry, the electric field lies on the plane normal to the cylinder axis, is angle-independent and radial, that is, normal to the cylinder’s surface. Its value, easily obtained using the electrostatic Gauss’s Law is⁹⁷:

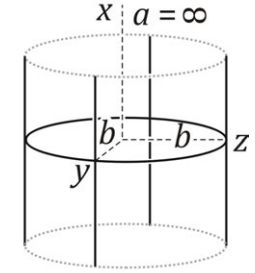


Figure 5. Infinitely long circular cylinder.

$$E_r = -\frac{\partial \phi(r)}{\partial r} = \begin{cases} 2\pi k_1 \rho \cdot r & \text{for } r \leq b \\ 2\pi k_1 \rho \frac{b^2}{r} & \text{for } r \geq b \end{cases}, \quad r = \sqrt{y^2 + z^2}. \quad (3.50)$$

Integrating the field with respect to r it is obtained

$$\phi(r) = \begin{cases} \text{constant} - \pi k_1 \rho \cdot r^2 & \text{for } r \leq b \\ \text{constant} - 2\pi k_1 \rho \cdot b^2 \ln r & \text{for } r \geq b \end{cases}, \quad (3.51)$$

$$f(\vec{r}) = \begin{cases} \frac{1}{4}(y^2 + z^2) & \text{for } \vec{r} \in V \\ \frac{1}{4}b^2 \ln(y^2 + z^2) & \text{for } \vec{r} \notin V \end{cases}. \quad (3.52)$$

From eq. 3.4 it follows that

⁹⁶ The reader should read in full detail the section *Conductors* in this book.

⁹⁷ A. Halpern, *3000 Solved Problems in Physics*, Schaum's Outline Series, McGraw-Hill, p. 406 problem 25.70; 1988.

$$\frac{\partial^2 f}{\partial x^2}=0, \quad \frac{\partial^2 f}{\partial y^2}=\frac{1}{2}, \quad \frac{\partial^2 f}{\partial z^2}=\frac{1}{2}, \quad \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}=0 \quad \text{for } x_\alpha \neq x_\beta, \quad r \in V, \quad (3.53)$$

$$\mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad (3.54)$$

which satisfies the trace rule eq. 3.15. This value of \mathbf{N} coincides, apart from the exchange of axis, with that obtained by symmetry arguments (see eqs. 3.25 and 3.73).

Outside the body

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial x_\alpha} &= \frac{\partial^2 f}{\partial x_\alpha \partial x} = 0 \quad \text{if } x_\alpha = y, z, \\ \frac{\partial^2 f}{\partial y^2} &= -\frac{b^2}{2} \frac{y^2 - z^2}{r^4}, \quad \frac{\partial^2 f}{\partial z^2} = \frac{b^2}{2} \frac{y^2 - z^2}{r^4}, \\ \frac{\partial^2 f}{\partial y \partial z} &= \frac{\partial^2 f}{\partial z \partial y} = -\frac{b^2}{2} \frac{2yz}{r^4}. \end{aligned} \quad (3.55)$$

In a spherical coordinate system where

$$x = r \cdot \cos \varphi \cdot \sin \theta, \quad y = r \cdot \sin \varphi \cdot \sin \theta, \quad z = r \cdot \cos \theta, \quad (3.56)$$

$$\delta = \sqrt{x^2 + y^2} = r \cdot \sin \theta,$$

and R is the cylinders's radius, the matrix expressions for \mathbf{n} are

Cylinder

$$\mathbf{N} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.57)$$

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \frac{R^2}{2\delta^2} \begin{pmatrix} -\cos 2\varphi & -\sin 2\varphi & 0 \\ -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.58)$$

As there are no components along the cylinder's z axis, the problem may be solved in the reduced space of the x, y components where the reduced inverse \mathbf{N}' is

$$\mathbf{N}' = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{N}'^{-1} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.59)$$

using which the general matrix equations may be applied for all electrostatic and magnetostatic cases.

Sphere

The sphere is the last body for which the electrostatic potential ϕ may be obtained from the electrostatic Gauss's Law. Let R be the radius of the sphere, Q the total uniformly distributed charge and take its center as the origin of coordinates. From the symmetry of the body the electric field has to be radial, $\vec{E}(\vec{r}) = E(r)\vec{r}$, the following function of the distance r to the center of the sphere.

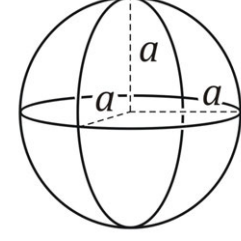


Figura 6. Sphere.

$$E(r) = \begin{cases} \frac{k_1 Q}{R^3} r & \text{for } r \leq R, \\ \frac{k_1 Q}{r^2} & \text{for } r > R. \end{cases} \quad (3.60)$$

As

$$\vec{E}(\vec{r}) = E(r)\hat{r} = -\nabla\phi(r) = -\frac{d}{dr}\phi(r), \text{ then } \phi(r) = -\int_{\infty}^r E(r')dr'. \quad (3.61)$$

Therefore

$$\phi(\vec{r}) = \begin{cases} \text{cte.} - k_1 \frac{Q}{2R^3} r^2 & \text{for } \vec{r} \in V \\ \text{cte.} + k_1 \frac{Q}{r} & \text{for } \vec{r} \notin V \end{cases}, \quad (3.62)$$

As

$$V = \frac{4\pi}{3}R^3, \quad \rho = \frac{Q}{V} = \frac{3Q}{4\pi R^3}, \quad f(r) = -\frac{1}{4\pi k_1 \rho} \phi(r) = -\frac{R^3}{3k_1 Q} \phi(r), \quad (3.63)$$

it follows that the auxiliary function f , eq. 3.8, is given by

$$f(\vec{r}) = \begin{cases} \frac{1}{6} r^2 & \text{for } r \leq R \\ -\frac{R^3}{3} \frac{1}{r} & \text{for } R \leq r \end{cases}. \quad (3.64)$$

From eq. 3.4 and

$$\frac{\partial^2 r^2}{\partial x_\alpha \partial x_\beta} = \frac{\partial^2 (x^2 + y^2 + z^2)}{\partial x_\alpha \partial x_\beta} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 r^{-1}}{\partial x_\alpha \partial x_\beta} = \frac{3x_\alpha x_\beta - \delta_{\alpha\beta} r^2}{r^5}, \quad (3.65)$$

where $\delta_{\alpha\beta}$ is Kronecker's delta, one gets the following expression.⁹⁸

⁹⁸ Soliv  rez (2008), p. 205 eq. 12.

$$\begin{aligned}
 \text{For the sphere } \mathbf{N} &= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \frac{1}{3} \mathbf{1}, \\
 \mathbf{n}^{\text{ext}}(\vec{r}) &= -\frac{R^3}{3} \begin{pmatrix} \frac{3x^2-r^2}{r^5} & \frac{3x \cdot y}{r^5} & \frac{3x \cdot z}{r^5} \\ \frac{3y \cdot x}{r^5} & \frac{3y^2-r^2}{r^5} & \frac{3y \cdot z}{r^5} \\ \frac{3z \cdot x}{r^5} & \frac{3z \cdot y}{r^5} & \frac{3z^2-r^2}{r^5} \end{pmatrix}.
 \end{aligned} \tag{3.66}$$

In dyadic notation

$$\mathbf{N} = \frac{1}{3} \mathbf{1}, \quad \mathbf{n}^{\text{ext}}(\vec{r}) = -\frac{R^3}{3} \frac{3\vec{r}\vec{r} - (\vec{r} \cdot \vec{r}) \mathbf{1}}{r^5} = -\frac{V}{4\pi} \frac{3\vec{r}\vec{r} - (\vec{r} \cdot \vec{r}) \mathbf{1}}{r^5}, \tag{3.67}$$

where V is the volume of the sphere. A general property of the depolarization tensor for finite bodies is here put into evidence: its internal value is independent of the body's volume V , while its external value is proportional to V . Notice that $\text{Tr } \mathbf{N} = 1$ and $\text{Tr } \mathbf{n}^{\text{ext}} = 0$, as previously discussed. Matrix \mathbf{N} coincides with that obtained in a simpler way from symmetry arguments (see [Problem 19](#)).

Other properties of the internal depolarization tensor \mathbf{N}

Integral expressions of the eigenvalues of \mathbf{N}

The proof of many important properties of the internal depolarization tensor \mathbf{N} may be easily derived from its general integral expressions. It is therefore convenient to use the values given by eqs. 4.6, as reproduced next. N_α and $N_\alpha(0)$ are different notations for the same integral. The values of $N_\alpha(\kappa)$ with $\kappa \neq 0$ are necessary for the calculation of the depolarization tensor external to the body.

$$N_a = \frac{abc}{2} \int_0^\infty \frac{ds}{(a^2+s)^{3/2} \sqrt{(b^2+s)(c^2+s)}}, \quad \text{where } a \geq b \geq c, \tag{3.68}$$

$$N_b = \frac{abc}{2} \int_0^\infty \frac{ds}{(b^2+s)^{3/2} \sqrt{(a^2+s)(c^2+s)}}, \quad \text{where } a \geq b \geq c, \tag{3.69}$$

$$N_c = \frac{abc}{2} \int_0^\infty \frac{ds}{(c^2+s)^{3/2} \sqrt{(a^2+s)(b^2+s)}}, \quad \text{where } a \geq b \geq c. \tag{3.70}$$

The integrals are also highly symmetric as regards the semiaxes and it may be easily proved that

$$N_b(a,b,c) = N_a(b,a,c), N_c(a,b,c) = N_a(c,b,a). \quad (3.71)$$

That is, one single function gives the three eigenvalues, as long as they are sorted in such a way that the condition $a \geq b \geq c$ is fulfilled. As they are very efficient methods for the numerical evaluation of integrals⁹⁹, eqs. 3.71 may simplify the numeral calculation of eigenvalues that are necessary for the general triaxial ellipsoids.

As the three integrals differ only in the single factor $1/(d_\alpha^2 + s)$ one might assume that

$$\text{if } a \geq b \geq c, \text{ then } \frac{1}{a^2 + s} \leq \frac{1}{b^2 + s} \leq \frac{1}{c^2 + s} \text{ and } N_a \leq N_b \leq N_c. \quad (3.72)$$

The conclusion is true, but the proof is flawed. This may be seen by using a similar argument to prove that for constant a and c , the eigenvalues of \mathbf{N} are a decreasing function of b . As seen from [Figure 14](#), [Figure 15](#) and [Figure 16](#), this is true for N_b , but not for N_a and N_c . The origin of the paradox is a combination of the symmetry properties eq. 3.71 and the constraints imposed on the relative values of a , b and c originated in the method used to derive the expressions of N_α in terms of elliptic functions. In this case the symmetries eqs. 3.71 are lost and each eigenvalue is given by a completely different function (see eqs. 4.9).

Diagonalization and inversion

As seen from eq. 4.6 the internal depolarization tensor \mathbf{N} is diagonal in the ellipsoid's principal system of coordinates, the one coinciding with its semiaxes. Furthermore, from the integrals that define its eigenvalues (the diagonal matrix elements or tensor's principal values) it is easily proved that they are never negative and do not vanish for finite values of the semiaxes. Therefore, the \mathbf{N} matrix of all finite ellipsoids is positive definite and always has an inverse¹⁰⁰ \mathbf{N}^{-1} . The only exception are the cases when one or two semiaxes become infinite, where the corresponding eigenvalues vanish, as discussed next. The calculation of the internal and external fields for these cases is discussed at sections [Sheet of constant thickness and infinite extension](#) and [Right circular cylinder of infinite length](#).

Infinite semiaxes and the inverse of \mathbf{N}

The constant thickness sheet of infinite extension and the infinitely long right circular cylinder are two cases often discussed in introductory courses of electromagnetism because of the simplicity of their treatment. They are not real bodies, but they provide useful illustration of important properties and the fields thus obtained are good approximations to those of a very long prolate spheroid near the equator (for the cylinder's case) and to a very short oblate spheroid near

⁹⁹ Press, W. H. & Teukolsky, S. A. & Vetterling, W. T. & Flannery, B. P.; *Numerical recipes in C*; Cambridge University Press (United Kingdom); 1992.

¹⁰⁰ Goertzels and Tralli, p. 16.

the axis (for the sheet's case). Both cases can be taken to be the limits of spheroidal ellipsoids where one axis (the symmetry one) or two axis (the equatorial ones) are infinitely long. A third case will also be considered, the infinite right elliptic cylinder, but only after the triaxial ellipsoids is discussed.

When semiaxis a becomes very large, the following limit is obtained for its integral definition eq. 3.68:

$$\begin{aligned} N_a(\kappa) &= \lim_{a \rightarrow \infty} \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(a^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \\ &= \lim_{a \rightarrow \infty} \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{\left(a^{4/3} + \frac{s}{a^{2/3}}\right)^{3/2} \sqrt{(b^2 + s)(c^2 + s)}} = 0. \end{aligned} \quad (3.73)$$

As the same thing happens for all coordinate axes, when the value of semiaxis $\alpha \rightarrow \infty$, the corresponding principal value N_α goes to 0. One should then check if the other two principal values, eqs. 3.69 and 3.70, remain finite, which in our example are N_b and N_c . For N_b ,

$$\begin{aligned} N_b(\kappa) &= \lim_{a \rightarrow \infty} \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \\ &= \lim_{a \rightarrow \infty} \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2 + s)^{3/2} \sqrt{\left(1 + \frac{s}{a^2}\right)(c^2 + s)}} = \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{\sqrt{c^2 + s} (b^2 + s)^{3/2}}, \end{aligned} \quad (3.74)$$

where the integrand is finite and non-vanishing.

The same happens for $N_c(\lambda)$,

$$\begin{aligned} N_c(\kappa) &= \lim_{a \rightarrow \infty} \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \\ &= \lim_{a \rightarrow \infty} \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s)^{3/2} \sqrt{\left(1 + \frac{s}{a^2}\right)(b^2 + s)}} = \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{\sqrt{b^2 + s} (c^2 + s)^{3/2}}. \end{aligned} \quad (3.75)$$

The last two cases correspond to the same type of integral, that will be solved when discussing the infinitely long elliptic cylinder in next chapter.

The case $b = c$ corresponds to the infinitely long right circular cylinder, where

$$N_b(\kappa) = N_c(\kappa) = \frac{b^2}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2 + s)^2} = -\frac{b^2}{2(b^2 + s)} \Big|_{\kappa}^{\infty} = \frac{1}{2} \frac{b^2}{(b^2 + \kappa)}. \quad (3.76)$$

The next case for analysis is the sheet of constant thickness and infinite extension. To that end consider an ellipsoid with finite semiaxis c , letting the other two grow without limit. In order to preserve their ratio their values are written as ka and kb , and the limit $k \rightarrow \infty$ is taken. Then

$$\begin{aligned}
 \lim_{k \rightarrow \infty} N_c(\kappa) &= \lim_{k \rightarrow \infty} \frac{(ka)(kb)c}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s)^{3/2} \sqrt{(k^2 a^2 + s)(k^2 b^2 + s)}} \\
 &= \lim_{k \rightarrow \infty} \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s)^{3/2} \sqrt{\left(a^2 + \frac{s}{k^2}\right)\left(b^2 + \frac{s}{k^2}\right)}} = \frac{c}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s)^{3/2}} \\
 &= -\frac{c}{\sqrt{c^2 + s}} \Big|_{\kappa}^{\infty} = \frac{c}{\sqrt{c^2 + \kappa}}.
 \end{aligned} \tag{3.77}$$

The infinite limit for the integral expressions for $N_x(\kappa)$ and $N_y(\kappa)$ is easily proved to vanish. The same result is obtained at page 54 by using electrostatic Gauss's Law, where the field outside the body is also calculated showing it to be independent of the ratio a/b , so that there is no elliptic sheet of infinite extensi  n (although there is a right elliptic cylinder of infinite length, see eqs. 4.89 and 4.89). The result is confirmed by taking appropriate limits in the cases of the elliptic cylinder eq. 4.16 and the oblate spheroid eq. 4.29 with aspect ratio $\gamma = 0$.

In all these cases the inverse of \mathbf{N} does not exist and one should solve the equations that relate the components N_{ω} the applied field and the total field inside the body, as illustrated for the sheet of constant thickness and infinite extension at page 55 and for the right circular cylinder of infinite length at page 57.

\mathbf{N} is invariant for similar ellipsoids

An important property of the internal depolarization tensor \mathbf{N} is its independence of the body's volume, of the problem's scale. This is not evident from a mere inspection of eq. 4.6. To prove the assertion consider a similar ellipsoid, one whose semiaxes a' , b' , c' differ from the original ones by a finite constant factor k : $a' = k \cdot a$, $b' = k \cdot b$, $c' = k \cdot c$. The principal values of this similar ellipsoid turn out to be

$$N_{\alpha}(\kappa | ka, kb, kc) = \frac{k^3 abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(k^2 a'^2 + s)^2 \sqrt{(k^2 a^2 + s)(k^2 b^2 + s)(k^2 c^2 + s)}}, \tag{3.78}$$

apparently different. A change of integration variable $s = k^2 s'$ does not modify the limits of integration when $\kappa = 0$ and gives the identity

$$\begin{aligned}
N(0|ka, kb, kc) &= \frac{k^3 abc}{2} \int_0^\infty \frac{k^2 ds'}{(k^2 d_\alpha^2 + k^2 s')^2 \sqrt{(k^2 a^2 + k^2 s')(k^2 b^2 + k^2 s')(k^2 c^2 + k^2 s')}} \\
&= \frac{abc}{2} \int_0^\infty \frac{ds'}{(d_\alpha^2 + s')^2 \sqrt{(a^2 + s')(b^2 + s')(c^2 + s')}} = N(0|a, b, c) \\
N'_\alpha &= \frac{k^3 abc}{2} \int_0^\infty \frac{k^2 ds}{(k^2 n_\alpha^2 + k^2 s') \sqrt{(k^2 a^2 + k^2 s')(k^2 b^2 + k^2 s')(k^2 c^2 + k^2 s')}} \\
&= \frac{abc}{2} \int_0^\infty \frac{ds}{(n_\alpha^2 + s') \sqrt{(a^2 + s')(b^2 + s')(c^2 + s')}} = N_\alpha.
\end{aligned} \tag{3.79}$$

It follows that

$$N_\alpha(0|a, b, c) = N_\alpha(0|k \cdot a, k \cdot b, k \cdot c), \text{ where } \alpha = x, y, z. \tag{3.80}$$

It should be stressed that the property does not apply for the integrals with $\kappa \neq 0$.

Therefore, the internal depolarization tensors of similar ellipsoids are equal. This is of importance for the discussion of cavities in ellipsoidal bodies made in page 116. It should be stressed here that the external depolarization tensor \mathbf{n}^{ext} does not have this property; in fact, its value is proportional to the body's volume V .

$N(0)$ is determined by aspect ratios

The principal values of $N(0)$ are not a function of the three semiaxes, but of the ratios of two of them to a third, their aspect ratios. This is a corollary of the previous property when factor k in eqs. 3.80 is taken to be the inverse of the largest semiaxis, in this book taken to be a .

$$\begin{aligned}
N_\alpha(0) &= \frac{1}{2} \frac{a}{a} \frac{b}{a} \frac{c}{a} \int_0^\infty \frac{ds}{\left((d_\alpha/a)^2 + s\right) \sqrt{\left((a/a)^2 + s\right) \left((b/a)^2 + s\right) \left((c/a)^2 + s\right)}} \\
&= \frac{1}{2} \beta \gamma \int_0^\infty \frac{ds}{(\delta_\alpha^2 + s) \sqrt{(1+s)(\beta^2 + s)(\gamma^2 + s)}}, \\
&\text{where } \alpha = x, y, z, \quad \beta = \frac{b}{a}, \gamma = \frac{c}{a}, \quad \delta_x = 1, \delta_y = \beta, \delta_z = \gamma.
\end{aligned} \tag{3.82}$$

This is the convention used in the tabulations of $N_\alpha(0)$ for the triaxial ellipsoid, the only type where the eigenvalues cannot be expressed in terms of elementary functions.

For further applications it is convenient to identify the values of aspect ratios β and γ for different categories of ellipsoids, as shown in Table 3, where

$$a \geq b \geq c, \quad 1 \geq \beta \geq \gamma. \quad (3.83)$$

a	b	c	β	γ	Type of ellipsoid
∞	∞	c	⁻¹⁰¹	0	sheet of constant thickness and infinite extension (see p. 54)
∞	b	b	0	0	right circular cylinder of infinite length (see p. 56)
∞	b	$c < b$	0	0	right elliptic cylinder of infinite length (see p. 58)
a	a	$c < a$	1	c/a	oblate spheroid (see p. 69)
a	$b < a$	b	b/a	b/a	prolate spheroid (see p. 74)
a	a	a	1	1	Sphere (see p. 58)
a	$b < a$	$c < b$	b/a	c/a	triaxial ellipsoid (see p. 65)

Table 3. Values of β and γ corresponding to different types of ellipsoids.

¹⁰¹ See eq. 3.77.

Chapter 4: The depolarization tensor: advanced treatment

Calculations of n from gravitational potentials

The gravitational potential of all ellipsoidal shapes have been solved both inside and outside the body. The formulas obtained are useful not only for the calculation of the depolarization tensor of the asymmetric triaxial ellipsoid but also for the following shapes:

- $a > b > c$: triaxial ellipsoid;
- $a \gg b > c$: right elliptic cylinder of infinite length;
- $a = b > c$: oblate spheroid;
- $a > b = c$: prolate spheroid.

The formulas given in the previous chapter for the infinite sheet, infinite cylinder and sphere provide a verification of the validity of the formulas that will be obtained next.

Triaxial ellipsoid

The triaxial ellipsoid is the shape where all the semiaxes are different. Its principal coordinate system —as in all cases— is the one such that axes x, y, z coincide with the ordered semiaxes

$$a > b > c. \quad (4.1)$$

As will be seen shortly, the components of the depolarization tensor \mathbf{n} are here given by elliptic integrals, non-elementary transcendental functions unfamiliar for many physicists and engineers. As their values are usually calculated from series expansions or taken from tables, these functions are usually expressed in terms of a few canonical or normal ones (Legendre's elliptic functions), two of which appear here. Its definitions and properties in the range of interest are given at [Appendix 9](#). Nevertheless, as pointed out in section [Integral expressions of the eigenvalues of \$N\$](#) , it might prove simpler to use the single integral eq. 3.68 for the numerical calculation of internal depolarization factors. For these reason the components of \mathbf{n} will be expressed in two different ways.

The calculation of $V(\vec{r})$, the internal gravitational potential of an homogeneous solid ellipsoid of mass density σ can be made using either ellipsoidal¹⁰² or

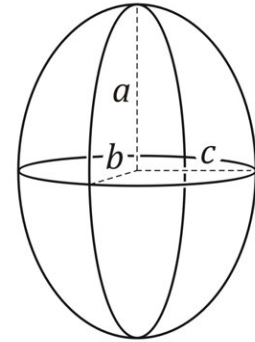


Figure 7. Triaxial ellipsoid.

¹⁰² Stratton, pp. 207-211.

cartesian coordinates¹⁰³. The latter, used here because it is the more concise approach, was made in 1839 by the french mathematician Gustave Lejeune Dirichlet¹⁰⁴ and widely reproduced afterwards in books on potential theory. The version of Dirichlet's proof given by MacMillan¹⁰⁵ is summarized next, where the gravitational constant was chosen by this author to be unity¹⁰⁶.

Taking as coordinate axes the principal ones of the ellipsoid, MacMillan changes to a system of coordinates with origin at field point \vec{r} . The radial part of the volume integral turns out to have an upper limit which is a function of \vec{r} . Using symmetry arguments, he shows that several of the resulting integrals vanish, leading to a quadratic expression in the field point coordinates,

$$V(\vec{r}) = \sigma \iiint_V \frac{d^3r'}{|\vec{r} - \vec{r}'|} = C_0 + C_x x^2 + C_y y^2 + C_z z^2. \quad (4.2)$$

The four coefficients are functions of the parameters a , b and c alone, and all may be expressed in terms of a single integral and its derivatives respect of the different semiaxes¹⁰⁷,

$$C_\alpha = \frac{1}{d_\alpha} \frac{\partial C_0}{\partial d_\alpha} - \frac{C_0}{d_\alpha^2}, \quad \text{where } \alpha = x, y, z, \quad d_x = a, d_y = b, d_z = c. \quad (4.3)$$

After a change of integration variable¹⁰⁸ more symmetric expressions for C_x , C_y , and C_z (see eq. 3.71) are obtained, as given below¹⁰⁹. As all these coefficients are needed to evaluate the external depolarization tensor \mathbf{n}^{ext} (see section [General expression of external \$\mathbf{n}\$](#)), it is convenient to use the auxiliary function f eq. 3.8:

$$f(\vec{r}) = -\frac{V(\vec{r})}{4\pi\sigma} = \frac{abc}{4} (-I_0 + I_x x^2 + I_y y^2 + I_z z^2), \quad (4.4)$$

where

$$I_0(0) = \int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \quad I_\alpha = \int_0^\infty \frac{ds}{(d_\alpha^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \quad (4.5)$$

where $\alpha = x, y, z$ and $d_x = a, d_y = b, d_z = c$.

¹⁰³ MacMillan, pp. 45-62.

¹⁰⁴ Stoner, p. 807.

¹⁰⁵ MacMillan, section 32, pp. 45-49.

¹⁰⁶ MacMillan, p. 24, eq. 20.1.

¹⁰⁷ MacMillan, p. 48 eq. 32.10.

¹⁰⁸ The substitution $\sin\varphi = c / \sqrt{c^2 + s}$ replaces the polar angle φ by the new integration variable s , and the upper limit of integration $\pi/2$ by ∞ .

¹⁰⁹ MacMillan, p. 49 eq. 32.13.

Notice that the notation $I_j(0)$ for the integrals comes about because it turns out that they are a special case of the integrals $I_j(\kappa)$ discussed in section [Obtention of the external gravitational potential by Ivory's method](#).

From eqs. 4.4, 4.5 and 3.4 the depolarization factors $N(0)$ are given by

$$N_\alpha(0) = \frac{1}{2}abc \int_0^\infty \frac{ds}{(d_\alpha^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}}, \quad (4.6)$$

where $\alpha = x, y, z$ and $d_x = a, d_y = b, d_z = c$.

The integrals may be checked for the depolarization tensor of the sphere ($a = b = c = R$), where

$$\begin{aligned} N_R &= \frac{1}{2}R^3 \int_0^\infty \frac{ds}{(R^2 + s) \sqrt{(R^2 + s)(R^2 + s)(R^2 + s)}} \\ &= \frac{1}{2}R^3 \int_0^\infty (R^2 + s)^{-5/2} ds = \frac{1}{2}R^3 \left[-\frac{2}{3}(R^2 + s)^{-3/2} \right]_0^\infty = \frac{1}{3}, \end{aligned} \quad (4.7)$$

coincident with the values previously derived both by symmetry arguments and by using Gauss's Law (eq. 3.66 and [Problem 19](#)).

Internal depolarization tensor N

Expressions with normal elliptic integrals E and F

The integrals eqs. 4.4 are reduced to Legendre's incomplete elliptic integrals of the first and second kind $F(\phi, k)$ and $E(\phi, k)$ (see [Appendix 9](#)) by the substitution

$$s = \frac{a^2}{\tan(t)^2} - \frac{c^2}{\sin(t)^2}. \quad (4.8)$$

It is thus obtained¹¹⁰:

$$\begin{aligned} N_x(0) &= N_a(0) = \frac{\beta \cdot \gamma}{(1 - \beta^2) \sqrt{1 - \gamma^2}} (-E(\phi, k) + F(\phi, k)), \\ N_y(0) &= N_b(0) = -\frac{\gamma^2}{\beta^2 - \gamma^2} + \frac{\beta \cdot \gamma \sqrt{1 - \gamma^2}}{(1 - \beta^2)(\beta^2 - \gamma^2)} E(\phi, k) - \frac{\beta \cdot \gamma}{(1 - \beta^2) \sqrt{1 - \gamma^2}} F(\phi, k), \\ N_z(0) &= N_c(0) = \frac{\beta^2}{(\beta^2 - \gamma^2)} - \frac{\beta \cdot \gamma}{(\beta^2 - \gamma^2) \sqrt{1 - \gamma^2}} E(\phi, k), \end{aligned} \quad (4.9)$$

¹¹⁰ MacMillan, pp. 49-50; Osborn, eqs. 2.1 to 2.6; Stoner, eqs. 3.5-.

$$\text{where } \beta = \frac{b}{a}, \gamma = \frac{c}{a}, k = \sqrt{\frac{1-\beta^2}{1-\gamma^2}}, \quad 0 \leq \phi = \cos^{-1} \gamma \leq \frac{\pi}{2}. \quad (4.10)$$

These principal values have the unit trace property eqs. 3.15. If one looks at the denominators, formulas seem to be valid only for

$$\beta \neq 1 (b \neq a), \beta \neq \gamma (b \neq c) \text{ and } \gamma \neq 1 (c \neq a).$$

This turns out to be false as may be seen from Figure 14, Figure 15 and Figure 16. The reason is that, when taking limits, expressions eqs. 4.9 converge to the right values.

N elliptic cylinder

The elliptic cylinder is the triaxial ellipsoid such that one semiaxis— a in our ordering—is extremely large. It includes, as a special case, the right circular cylinder of infinite length ($b = c$ in our convention). From symmetry arguments the properties of the cylinder are independent of coordinate x , and the system reduces to an apparently two dimensional one. Thus—as follows also from taking the limit $a \rightarrow \infty$ in eq. A7.2—the equation of the body’s surface is

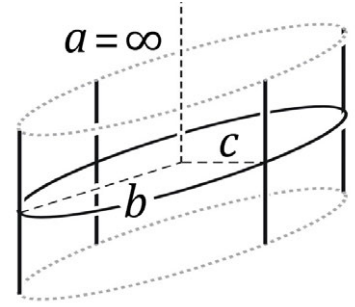


Figure 8. Elliptic cylinder.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (4.11)$$

The expressions of $N_\alpha(\kappa)$ ¹¹¹ is given by eqs. 3.73,

$$N_a^{\text{ec}}(\kappa) = 0. \quad (4.12)$$

The two other terms are given by eqs. 3.74 and 3.75, where the value for N_c may be obtained from that of N_b by permuting a with b . Solving the integral it is obtained¹¹²

$$\begin{aligned} N_b^{\text{ec}}(\kappa) &= \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2+s)^{3/2} \sqrt{c^2+s}} = \frac{bc}{2} \frac{2}{b^2-c^2} \frac{\sqrt{c^2+s}}{\sqrt{b^2+s}} \Bigg|_{\kappa}^{\infty} \\ &= \frac{bc}{b^2-c^2} \left(1 - \frac{\sqrt{c^2+\kappa}}{\sqrt{b^2+\kappa}} \right). \end{aligned} \quad (4.13)$$

¹¹¹ All integrals $I_\alpha(\kappa)$ are evaluated for $\kappa \neq 0$ because otherwise the calculations will have to be duplicated in the section devoted to the calculation of the *External depolarization tensor*.

¹¹² Korn & Korn, p. 941, eq. 148.

Permuting b and c gives

$$\begin{aligned} N_a^{\text{ec}}(\kappa) &= 0, & N_b^{\text{ec}}(\kappa) &= \frac{bc}{b^2 - c^2} \left(\frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{b^2 + \kappa}} \right), \\ N_c^{\text{ec}}(\kappa) &= \frac{bc}{b^2 - c^2} \left(\frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{c^2 + \kappa}} \right). \end{aligned} \quad (4.14)$$

The depolarization factors are obtained¹¹³ by setting $\kappa = 0$ in the previous equations, giving

$$\boxed{N_a^{\text{ec}}(0) = 0, \quad N_b^{\text{ce}}(0) = \frac{c}{b+c}, \quad N_c^{\text{ce}}(0) = \frac{b}{b+c}.} \quad (4.15)$$

The values satisfy the unit trace rule eq. 3.15 and the order $N_a \leq N_b \leq N_c$. Notice that these eigenvalues cannot be obtained from the graphs given by Figure 14, Figure 15 or Figure 16.

For the right circular cylinder the last two eigenvalues reduce to the value $\frac{1}{2}$ given by eq. 3.57. From here one may obtain the constant thickness sheet of infinite extension. The limits to take are

$$\begin{aligned} N_b^{\text{cts}}(0) &= \lim_{b \rightarrow \infty} N_b^{\text{ce}}(0) = \lim_{b \rightarrow \infty} \frac{c}{b+c} = 0, \\ N_c^{\text{cts}}(0) &= \lim_{b \rightarrow \infty} N_c^{\text{ce}}(0) = \lim_{b \rightarrow \infty} \frac{b}{b+c} = \lim_{b \rightarrow \infty} \frac{1}{1 + \frac{c}{b}} = 1, \end{aligned} \quad (4.16)$$

which coincide with the calculations made using Gauss's Theorem, eq. 3.47.

N oblate spheroid

An oblate spheroid is, in our convention, the ellipsoid of revolution such that its polar semiaxis c , its axis of rotational symmetry, is smaller than the equatorial ones: $a = b > c$. The equatorial eigenvalue N_e of the oblate spheroid is obtained from eq. 4.6. As the same integrals will be used later for the expression of the external tensor, the lower limit κ is taken, that for the internal case should be made zero. Thus,

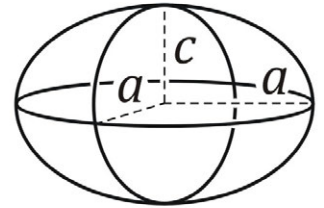


Figure 9. Oblate spheroid.

¹¹³ Osborn, eqs. 2.17 and 2.18. MacMillan, p. 71, use value of V with $\kappa = 0$.

$$N_e^o(\kappa) = N_a^o(\kappa) = N_b^o(\kappa) = \frac{a^2 c}{2} \int_{\kappa}^{\infty} \frac{ds}{\sqrt{c^2 + s} (a^2 + s)^2}, \quad (4.17)$$

The primitive of this integral is given in [Appendix 8](#), the constants being $A = c^2$, $B = a^2$, $A < B$. Therefore, eq. [A8.5](#) applies giving the following equatorial term.

$$\begin{aligned} N_e^o(\kappa) &= \frac{a^2 c}{2} \left(\frac{\sqrt{c^2 + s}}{(a^2 - c^2)(a^2 + s)} + \frac{1}{(a^2 - c^2)^{3/2}} \arctan \sqrt{\frac{c^2 + s}{a^2 - c^2}} \right) \Bigg|_{\kappa}^{\infty} \\ &= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \left(\pi/2 - \arctan \sqrt{\frac{c^2 + \kappa}{a^2 - c^2}} \right). \end{aligned} \quad (4.18)$$

Using

$$\tan \left(\frac{\pi}{2} - \varphi_0 \right) = \frac{1}{\tan \varphi_0} = \sqrt{\frac{a^2 - c^2}{c^2 + \kappa}} = \tan \varphi, \quad (4.19)$$

$$\begin{aligned} N_e^o(\kappa) &= N_a^o(\kappa) = N_b^o(\kappa) \\ &= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan \left(\frac{\sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}} \right). \end{aligned} \quad (4.20)$$

Alternative expressions for angle φ can be obtained as follows¹¹⁴.

$$\begin{aligned} \tan \varphi &= \frac{\sin \varphi}{\cos \varphi}, \quad \sin \varphi = k \sqrt{a^2 - c^2}, \quad \cos \varphi = k \sqrt{c^2 + \kappa}, \\ \sin^2 \varphi + \cos^2 \varphi &= k^2 (a^2 + \kappa) = 1, \quad \text{so that } k = (a^2 + \kappa)^{-1/2}, \\ \text{therefore } \sin \varphi &= \sqrt{\frac{a^2 - c^2}{a^2 + \kappa}} < 1, \quad \cos \varphi = \sqrt{\frac{c^2 + \kappa}{a^2 + \kappa}} < 1. \end{aligned} \quad (4.21)$$

¹¹⁴ MacMillan, p. 62 eq. 39.2, uses the expression in terms of arcsin.

$$\begin{aligned}
N_e^o(\kappa) &= N_a^o(\kappa) = N_b^o(\kappa) \\
&= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan \sqrt{\frac{a^2 - c^2}{c^2 + \kappa}} \\
&= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arccos \sqrt{\frac{c^2 + \kappa}{a^2 + \kappa}} \\
&= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arcsin \sqrt{\frac{a^2 - c^2}{a^2 + \kappa}}.
\end{aligned} \tag{4.22}$$

For $\kappa = 0$, the eigenvalues of \mathbf{N} may be expressed in terms of the aspect ratio γ as follows. Notice that $\beta = \gamma$, so that either one may be used in the equation. The second is chosen so that [Figure 10](#) and [Figure 12](#) have the same independent variable.

$$\begin{aligned}
N_e^o(0) &= N_a^o(0) = N_b^o(0) \\
&= -\frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)} + \frac{1}{2} \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arctan \left(\frac{\sqrt{1 - \gamma^2}}{\gamma} \right) \\
&= -\frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)} + \frac{1}{2} \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arccos(\gamma) \\
&= -\frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)} + \frac{1}{2} \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arcsin(\sqrt{1 - \gamma^2}).
\end{aligned} \tag{4.23}$$

Osborn and Maxwell use the expression in terms of \arcsin ¹¹⁵ instead of the simpler one in terms of \arccos .

The polar term may be obtained solving the definition integral [eq. 4.6](#), whose primitive is given in [Appendix 8](#), where the constants take the values $A = c^2$, $B = 1$ so that $A < B$. Therefore, [eq. A8.11](#) applies giving

¹¹⁵ Osborn, eq. 2.19. Maxwell, p. 69 eqs. 438.11 and 438.12, where $N_p = -L/4\pi$, $N_e = -M/4\pi = -N/4\pi$, $e^2 = 1 - \gamma^2$.

$$\begin{aligned}
N_p^o(\kappa) &= N_c^o(\kappa) = \frac{a^2 c}{2} \int_{\kappa}^{\infty} \frac{ds}{(c^2 + s)^{3/2} (a^2 + s)}, \\
&= \left(-\frac{a^2 c}{\sqrt{c^2 + s} (a^2 + s)} - \frac{a^2 c \sqrt{c^2 + s}}{(a^2 - c^2)(a^2 + s)} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan\left(\frac{\sqrt{c^2 + s}}{\sqrt{a^2 - c^2}}\right) \right) \Bigg|_{\kappa}^{\infty} \\
&= \frac{a^2 c}{(a^2 - c^2) \sqrt{c^2 + \kappa}} + \frac{a^2 c}{(a^2 - c^2)^{3/2}} \left[\arctan\left(\frac{\sqrt{c^2 + \kappa}}{\sqrt{a^2 - c^2}}\right) - \frac{\pi}{2} \right].
\end{aligned} \tag{4.24}$$

Using

$$\tan(\varphi - \pi/2) = -1/\tan(\varphi_0), \tag{4.25}$$

and eqs. 4.21, the following expressions are obtained for $N_{fc}(\kappa)$.

$$\begin{aligned}
N_p^o(\kappa) &= N_c^o(\kappa) \\
&= \frac{a^2 c}{(a^2 - c^2) \sqrt{c^2 + \kappa}} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan \sqrt{\frac{a^2 - c^2}{c^2 + \kappa}} \\
&= \frac{a^2 c}{(a^2 - c^2) \sqrt{c^2 + \kappa}} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arccos \sqrt{\frac{c^2 + \kappa}{a^2 + \kappa}} \\
&= \frac{a^2 c}{(a^2 - c^2) \sqrt{c^2 + \kappa}} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arcsin \sqrt{\frac{a^2 - c^2}{a^2 + \kappa}}.
\end{aligned} \tag{4.26}$$

For $\kappa = 0$, this eigenvalue of N may be expressed in terms of the aspect ratio γ as follows:

$$\begin{aligned}
N_p^o(0) &= N_c^o(0) \\
&= \frac{1}{(1 - \gamma^2)} - \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arctan \frac{\sqrt{1 - \gamma^2}}{\gamma} \\
&= \frac{1}{1 - \gamma^2} - \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arccos(\gamma) \\
&= \frac{1}{1 - \gamma^2} - \frac{\gamma}{(1 - \gamma^2)^{3/2}} \arcsin(\sqrt{1 - \gamma^2}).
\end{aligned} \tag{4.27}$$

Notice, by inspection, that eqs. 4.23 and 4.27 fulfill the unit trace rule

$$\text{Tr } \mathbf{N} = 2N_e^0 + N_p^0 = 1. \quad (4.28)$$

If $\gamma \ll 1$ (very flat oblate spheroid), the eigenvalues may be approximated by¹¹⁶

$$N_e = N_a = N_b \cong \frac{\pi}{4}\gamma - \gamma^2, \quad N_p = N_c \cong 1 - \frac{\pi}{2}\gamma + 2\gamma^2, \quad (4.29)$$

which should be compared with the values given at section *Sheet of constant thickness and infinite extension*.

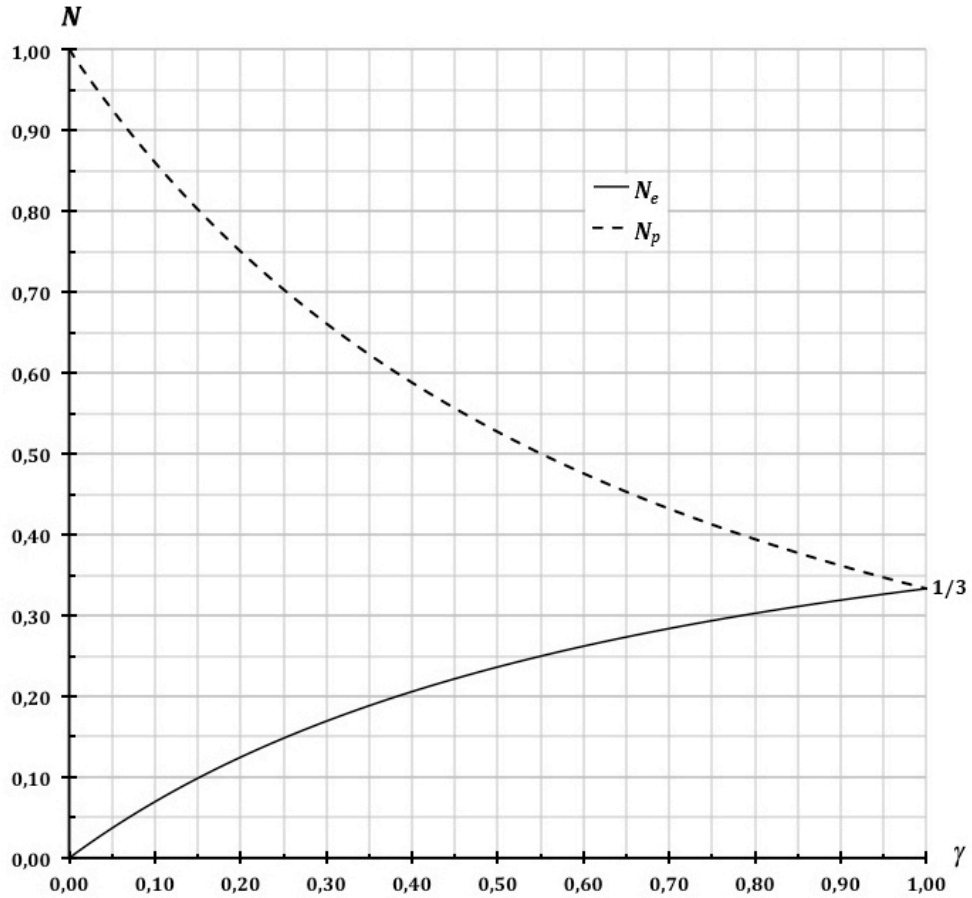


Figure 10. N_p and N_e as a function of γ for the oblate spheroid.

Figure 10 shows the values of the equatorial eigenvalue N_e and the polar one N_p for the oblate spheroid. The values $\beta = \gamma = 0$ ($a = b = \infty$) correspond to the constant thickness sheet of infinite extension where $N_p = 1$, $N_e = 0$ (Table 3 and eq. 3.47); the values of the sphere are $\beta = \gamma = 1$ (eq. 3.66). Both functions will turn out to be important in the discussion of the eigenvalues of \mathbf{N} for the triaxial ellipsoid (see section *Graphs of N_a , N_b and N_c*) where they provide upper or lower bounds of certain eigenvalues.

¹¹⁶ Osborn, eq. 2.21.

N prolate spheroid

A prolate spheroid is an ellipsoid of revolution (spheroid) such that—in our convention—the polar semiaxis N_a , its axis of rotational symmetry, is greater than the equatorial ones: $a > b = c$.

The equatorial eigenvalues of **N** for the prolate spheroid are obtained from eq. 4.6. As the same integrals will be used later for the expression of the external tensor, the lower limit κ is taken, that for the internal case should be made zero. Using the upper index “p” to identify the prolate spheroid, it is found that

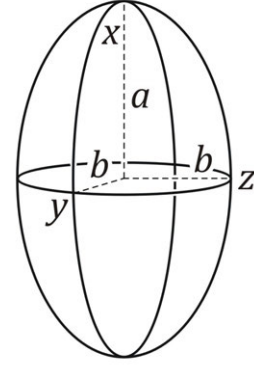


Figure 11. Prolate spheroid.

$$N_e^p(\kappa) = N_b^p(\kappa) = N_c^p(\kappa) = \frac{ac^2}{2} \int_{\kappa}^{\infty} \frac{ds}{\sqrt{(a^2+s)(c^2+s)}^2}. \quad (4.30)$$

The integral eq. 4.30 is discussed in Appendix 8, where the constants are $A = a^2$, $B = c^2$, $A > B$, so that eq. A8.6 applies. It is thus found

$$\begin{aligned} N_e^p(\kappa) &= N_b^p(\kappa) = N_c^p(\kappa) \\ &= \left[-\frac{1}{2} \frac{\gamma^2 \sqrt{1+s}}{(1-\gamma^2)(\gamma^2+s)} - \frac{1}{4} \frac{\gamma^2}{(1-\gamma^2)^{3/2}} \ln \left(\frac{\sqrt{1+s} - \sqrt{1-\gamma^2}}{\sqrt{1+s} + \sqrt{1-\gamma^2}} \right) \right]_{\kappa}^{\infty} \\ &= \frac{1}{2} \frac{ac^2 \sqrt{a^2+\kappa}}{(a^2-c^2)(c^2+\kappa)} + \frac{1}{4} \frac{ac^2}{(a^2-c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2+\kappa} - \sqrt{a^2-c^2}}{\sqrt{a^2+\kappa} + \sqrt{a^2-c^2}} \right). \end{aligned} \quad (4.31)$$

The value of the logarithm can be made positive (argument > 1) and this function can be replaced by the hyperbolic ones by using the some of the following identities:

$$\ln \left(\frac{\sqrt{a^2+\kappa} - \sqrt{a^2-c^2}}{\sqrt{a^2+\kappa} + \sqrt{a^2-c^2}} \right) = -\ln \left(\frac{\sqrt{a^2+\kappa} + \sqrt{a^2-c^2}}{\sqrt{a^2+\kappa} - \sqrt{a^2-c^2}} \right) = -2 \ln \left(\frac{(\sqrt{a^2+\kappa} + \sqrt{a^2-c^2})}{\sqrt{c^2+\kappa}} \right). \quad (4.32)$$

The following relationship¹¹⁷ is used to obtain alternative expressions:

$$\begin{aligned} \text{if } \cosh \alpha &= \sqrt{\frac{a^2 + \kappa}{c^2 + \kappa}}, \\ e^\alpha &= \cosh \alpha + \sinh \alpha = \cosh \alpha + \sqrt{\cosh^2 \alpha - 1} = \frac{(\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2})}{\sqrt{c^2 + \kappa}}. \end{aligned} \quad (4.33)$$

$$\begin{aligned} N_e^p(\kappa) &= N_b^p(\kappa) = N_c^p(\kappa) \\ &= \frac{1}{2} \frac{ac^2 \sqrt{a^2 + \kappa}}{(a^2 - c^2)(c^2 + \kappa)} - \frac{1}{4} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{a^2 + \kappa} - \sqrt{a^2 - c^2}} \right) \\ &= \frac{1}{2} \frac{ac^2 \sqrt{a^2 + \kappa}}{(a^2 - c^2)(c^2 + \kappa)} - \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}} \right) \\ &= \frac{1}{2} \frac{ac^2 \sqrt{a^2 + \kappa}}{(a^2 - c^2)(c^2 + \kappa)} - \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \cosh^{-1} \sqrt{\frac{a^2 + \kappa}{c^2 + \kappa}}. \end{aligned} \quad (4.34)$$

The corresponding eigenvalues of \mathbf{N} are found setting $\kappa = 0$ in eqs. 4.34 and using the aspect ratio γ . Notice that $\beta = \gamma$, so that either one may be used in the equation, and the second is chosen so that Figure 10 and Figure 12 have the same independent variable¹¹⁸. Then

$$\begin{aligned} N_e^p(0) &= N_b^p(0) = N_c^p(0) \\ &= \frac{1}{2} \frac{1}{(1 - \gamma^2)} - \frac{1}{4} \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \ln \left(\frac{1 + \sqrt{1 - \gamma^2}}{1 - \sqrt{1 - \gamma^2}} \right) \\ &= \frac{1}{2} \frac{1}{(1 - \gamma^2)} - \frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \ln \left(\frac{(1 + \sqrt{1 - \gamma^2})}{\gamma} \right) \\ &= \frac{1}{2} \frac{1}{(1 - \gamma^2)} - \frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \cosh^{-1} \left(\frac{1}{\gamma} \right). \end{aligned} \quad (4.35)$$

¹¹⁷ Korn & Korn, p. 817 eqs. 21.2-37

¹¹⁸ Osborn, eq. 2.11. Macmillan, p. 63 eq. 4 with $\kappa = 0$.

The integral expression for the polar term may be obtained from eq. 4.6 giving

$$N_p^p(\kappa) = N_a^p(\kappa) = \frac{ac^2}{2} \int_{\kappa}^{\infty} \frac{ds}{(a^2 + s)^{3/2} (c^2 + s)} \quad (4.36)$$

The primitive of this integral is given in [Appendix 8](#). As $A = a^2$, $B = c^2$, $A > B$, eq. A8.12 should be used obtaining

$$\begin{aligned} N_p^p(\kappa) = N_a^p(\kappa) &= \left(\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + s}} + \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + s} - \sqrt{a^2 - c^2}}{\sqrt{a^2 + s} + \sqrt{a^2 - c^2}} \right) \right) \Bigg|_{\kappa}^{\infty} \\ &= -\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + \kappa}} + \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{a^2 + \kappa} - \sqrt{a^2 - c^2}} \right). \end{aligned} \quad (4.37)$$

where the argument has been inverted to make positive the value of the natural logarithm. The same alternatives as for eqs. 4.34 are valid here, giving

$$\begin{aligned} N_p^p(\kappa) &= N_a^p(\kappa) \\ &= -\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + \kappa}} + \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{a^2 + \kappa} - \sqrt{a^2 - c^2}} \right) \\ &= -\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + \kappa}} + \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}} \right) \\ &= -\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + \kappa}} + \frac{ac^2}{(a^2 - c^2)^{3/2}} \cosh^{-1} \sqrt{\frac{a^2 + \kappa}{c^2 + \kappa}}. \end{aligned} \quad (4.38)$$

The eigenvalues of $\mathbf{N}(0)$ in terms of aspect ratio γ are

$$\begin{aligned} N_p^p(0) &= N_c^p(0) \\ &= -\frac{\gamma^2}{(1 - \gamma^2)} + \frac{1}{2} \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \ln \left(\frac{1 + \sqrt{1 - \gamma^2}}{1 - \sqrt{1 - \gamma^2}} \right) \\ &= -\frac{\gamma^2}{(1 - \gamma^2)} + \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \ln \left(\frac{1 + \sqrt{1 - \gamma^2}}{\gamma} \right) \\ &= -\frac{\gamma^2}{(1 - \gamma^2)} + \frac{\gamma^2}{(1 - \gamma^2)^{3/2}} \cosh^{-1} \left(\frac{1}{\gamma} \right). \end{aligned} \quad (4.39)$$

For very slender prolate spheroids the following approximations are valid¹¹⁹,

$$\begin{aligned} N_e(0) = N_b(0) = N_c(0) &\cong \frac{1}{2}(1 + \gamma^2) - \frac{1}{2}\gamma^2 \ln(2/\gamma), \\ N_p(0) = N_a(0) &\cong -\gamma^2 + \gamma^2 \ln(2/\gamma), \quad \text{if } \gamma \ll 1. \end{aligned} \quad (4.40)$$

The aspect ratio γ vanishes in the limit of infinitely large a , whereupon one gets the eigenvalues of the right circular clinder of infinite length previously obtained by using the electrostatic Gauss's Law (see eqs. 3.57).

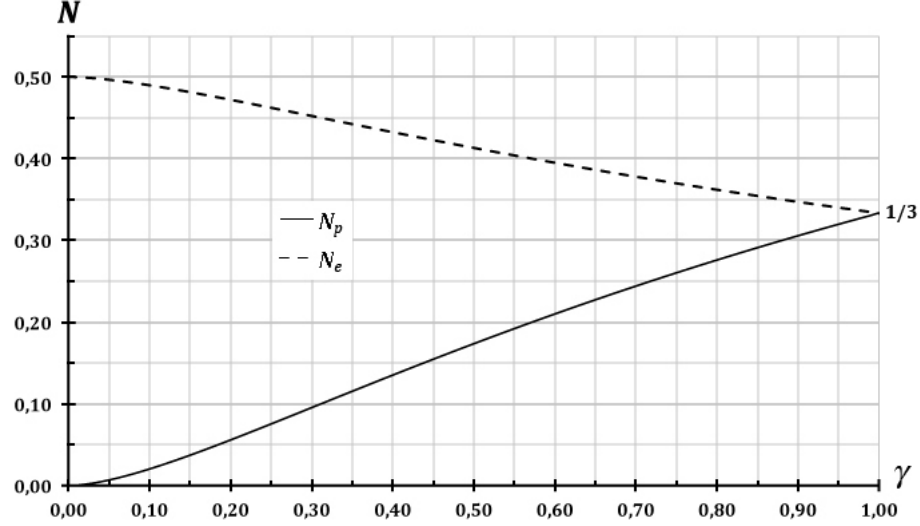


Figure 12. N_p and N_e as a function of γ for the prolate spheroid.

Figure 12 shows the values of the polar eigenvalue N_p and the equatorial one N_e for the prolate spheroid. The values $\gamma = 0$ ($a = \infty$), $N_p = 0$ correspond to the infinite right circular cylinder where $N_e = 1/2$ (Table 3 and eq. 3.66); the values $\beta = \gamma = 1$, $N_a = N_b = N_c = 1/3$, to the sphere (eq. 3.66). Both functions will turn out to be important in the discussion of the eigenvalues of \mathbf{N} for the triaxial ellipsoid (see section *Graphs of N_a , N_b and N_c*) where they provide upper or lower bounds.

Unified treatment of N for spheroids

As pointed out in section *Integral expressions of the eigenvalues of N* , a single integral like eq. 3.68 characterizes all eigenvalues of \mathbf{N} . The same thing happens with more specialized integrals, like the ones with two equal semiaxes that give the spheroid's eigenvalues. The reason for having different integrals for N_a , N_b , N_c is the convention $a \geq b \geq c$, which underlies eq. 4.51. This subtle point is discussed next.

Consider the integral expression for the equatorial eigenvalue of the oblate spheroid eq. 4.17, where the domain of positive real numbers is divided in two,

¹¹⁹ Osborn, eq. 2.13.

$$\begin{aligned}
N_e^o &= \frac{1}{2} \gamma \int_0^\infty \frac{ds}{(1+s)^2 \sqrt{\gamma^2 + s}}, \quad \text{where } 0 \leq \gamma = \frac{c}{a} \leq 1, \\
N^? &= \frac{1}{2} \delta \int_0^\infty \frac{ds}{(1+s)^2 \sqrt{\delta^2 + s}}, \quad \text{where } 1 \leq \delta = \frac{c}{a}.
\end{aligned}
\tag{4.41}$$

The added ranges are expected to give a single continuous curve with continuous slope. The curve for $0 \leq \gamma \leq 1$ —the same given in [Figure 10](#)—is the dashed one in the left half of [Figure 13](#). $N^?(\delta)$, not represented there, is an increasing function of δ with minimum value $1/3$ at $\delta = 1$ and maximum value $1/2$ at $\delta = \infty$. As that domain of the independent variable may not be represented directly, it is convenient to make a change of variable to $1/\delta$, that corresponds to the domain $[1, 0]$, its graph being the dashed curve on the right side of [Figure 13](#). This resulting integral is

$$\begin{aligned}
N^? &= \frac{1}{2} \frac{1}{\delta} \int_0^\infty \frac{ds}{(1+s)^2 \sqrt{\frac{1}{\delta^2} + s}} = \frac{1}{2} \int_0^\infty \frac{\delta^4 ds}{(\delta^2 + \delta^2 s)^2 \sqrt{1 + \delta^2 s}} \\
&= \frac{1}{2} \int_0^\infty \frac{\delta^2 ds'}{(\delta^2 + s')^2 \sqrt{1 + s'}}, \quad \text{where } s' = \delta^2 s, \quad 1 \geq \frac{1}{\delta} = \frac{a}{c} \geq 0.
\end{aligned}
\tag{4.42}$$

The last integral coincides with that for the equatorial eigenvalue of the prolate spheroid, eq. 4.30. Its graphic—previously given in [Figure 12](#)—is the dashed line at the right of [Figure 13](#). The definition $\delta = a/c$ seems to contradict its identification with the parameter $\gamma = c/a$. As it was initially assumed that $c > a$, by the convention $a \geq b \geq c$ the exchange of names $a \leftrightarrow c$ should be made. That is why in the right side of [Figure 13](#) the independent variable, that does not respect the convention, is labeled as a'/c' .

Therefore

$$N^? = \frac{1}{2} \gamma^2 \int_0^\infty \frac{ds}{(\gamma^2 + s)^2 \sqrt{1 + s}} = N_e^p
\tag{4.43}$$

is the equatorial eigenvalue of the prolate spheroid, eq. 4.30. The reader should verify, in a similar fashion, that the extension to $\gamma > 1$ of the expression of the polar eigenvalue of the oblate spheroid gives the polar eigenvalue of the prolate spheroid, the continuous line in [Figure 13](#).

The functions that give the dependence of the eigenvalues is very different for the oblate (inverse trigonometric functions) and the prolate spheroid (logarithm or inverse hyperbolic functions), as seen in eqs. A8.5 and A8.6. The origin of the difference is the sign of $\Delta = \pm(1 - \gamma^2)$, that is, whether c is smaller or greater than a .

MacMillan¹²⁰ has shown that the use of complex variable unifies both solutions (see [Problem 26](#))

Graphs of N_a , N_b and N_c

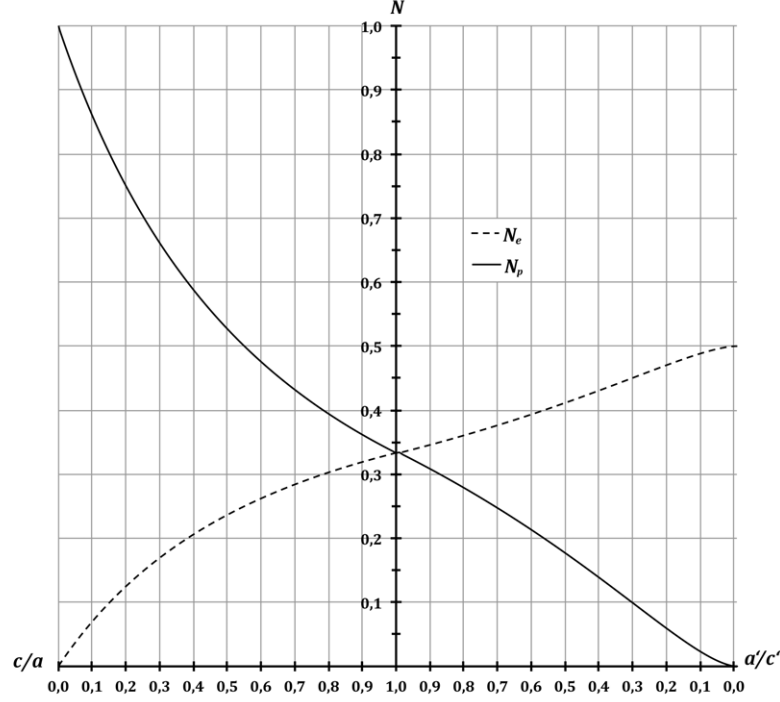


Figure 13. The equatorial and polar eigenvalues of spheroids are continuous curves.

For accurate calculations of the eigenvalues of N the best choice is to use mathematical software that includes the evaluation of the incomplete elliptic functions $E(\phi, k)$ and $F(\phi, k)$. On the other hand, a better understanding of their behavior is obtained by the use of graphs. As discussed in section [N\(0\) is determined by aspect ratios](#), the eigenvalues are functions of the ratios of semiaxes $\beta = b/a$ and $\gamma = c/a$. The best graphs known by the author, those of Osborn, depict them as sets of functions of γ , where each of the 11 curves has fixed values of $0 \leq \beta \leq 1$ at intervals of 0.1. These graphs for N_a , N_b and N_c are reproduced next at full page width.

As shown in the combined graph for the three eigenvalues, [Figure 17](#), for any fixed value of γ the eigenvalues are ordered as follows:

$$N_a \leq N_b \leq N_c. \quad (4.44)$$

Two finite eigenvalues may be equal only for the equatorial case of the prolate and oblate spheroids (see [Figure 12](#) and [Figure 10](#)). The lowest values of N_a are given by the curve for the polar eigenvalues of the prolate spheroids (see [Figure 12](#)) and the largest values of $N_c(\gamma, \gamma)$ for the polar eigenvalues of the oblate spheroids (see

¹²⁰ MacMillan, p. 63 eq. 39.4.

Figure 10). The in-between case N_b is bounded from below by N_a and from above by N_c .

N_a

Function $N_a(\beta, \gamma)$ is shown in Figure 14, where its values are confined to the grey shaded region, bounded by the upper curve $N_a(1, \gamma)$ (oblate spheroid, see Figure 10) and the lowest curve $N_a(\gamma, \gamma)$ (prolate spheroid, see Figure 12) as follows from Table 3 the ordering of semiaxes convention eq. 3.83. Therefore

$$0 \leq N_a(\gamma_0, \gamma_0) \leq N_a(\beta, \gamma_0) \leq N_a(1, \gamma_0) \leq 1/3. \quad (4.45)$$

For fixed γ_0 , as shown by the constant β curves, N_a is an increasing function of β ; that is

$$\text{if } \beta_2 > \beta_1, N_a(\beta_2, \gamma_0) > N_a(\beta_1, \gamma_0). \quad (4.46)$$

Each of the curves drawn correspond to a different value of the parameter $\beta = b/a$, the highest being the one for the oblate spheroid. The upper vertex $N_a(1, 1) = N_b(1, 1) = N_c(1, 1) = 1/3$ corresponds to the sphere, the transition value from oblate to prolate spheroids. The lowest vertex, $N_a(\beta, 0) = N_a(1, 0) = 0$, characterizes the constant thickness sheet of infinite extension (see Table 3).

N_b

Function $N_b(\beta, \gamma)$ is shown in Figure 15, where its values are confined to the grey shaded region, bounded by the lower curve $N_b(1, \gamma)$ (oblate spheroid, see Figure 10) and the upper curve $N_a(\gamma, \gamma)$ (prolate spheroid, see Figure 12) as follows from Table 3 and the ordering of semiaxes convention eq. 3.83. Therefore

$$0 \leq N_b(1, \gamma_0) \leq N_b(\beta, \gamma_0) \leq N_b(\gamma_0, \gamma_0) \leq 1/2. \quad (4.47)$$

For fixed γ_0 , as shown by the constant β curves, N_b is an decreasing function of β ; that is

$$\text{if } \beta_2 > \beta_1, N_b(\beta_2, \gamma_0) < N_b(\beta_1, \gamma_0). \quad (4.48)$$

Each of the curves drawn corresponds to a different value of the parameter $\beta = b/a$, the highest being the one for the prolate spheroid. The right hand vertex $N_b(1, 1) = N_a(1, 1) = N_c(1, 1) = 1/3$ corresponds to the sphere, the common limit of the oblate and prolate spheroids. The lowest left vertex, $N_b(\beta, 0) = N_b(1, 0) = 0$, characterizes the constant thickness sheet of infinite extension; the upper left vertex, $N_b(0, 0)$, characterizes the right elliptic cylinder of infinite length (see Table 3). The vertical line $\beta = 0$ corresponds to the the elliptic cylinders, whose values cannot be identified from the graph, because all of them have $\beta = \gamma = 0$ in this parametrization.

N_c

Function $N_c(\beta, \gamma)$ is shown in Figure 16, where its values are confined to the grey shaded region, bounded by the upper curve $N_c(1, \gamma)$ (oblate spheroid, see Figure 10) and the lowest curve $N_a(\gamma, \gamma)$ (prolate spheroid, see Figure 12) as follows from Table 3 and the ordering of semiaxes convention eq. 3.83.

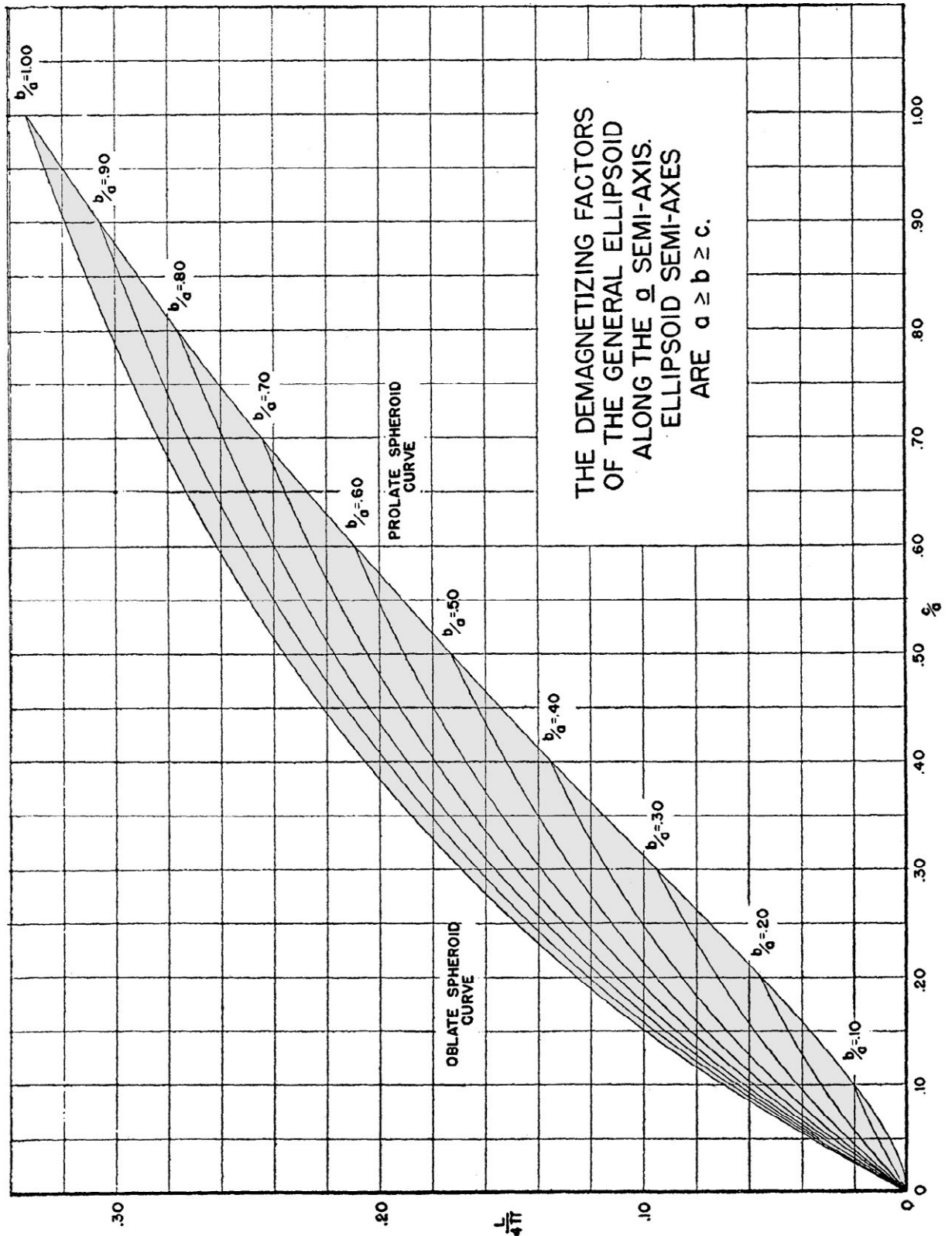


Figure 14. $N_a (L/4\pi)$ as a function of $c/a (\gamma)$, where each curve corresponds to a different value of the parameter $b/a (\beta)$ (taken from Osborn, Figure 1).

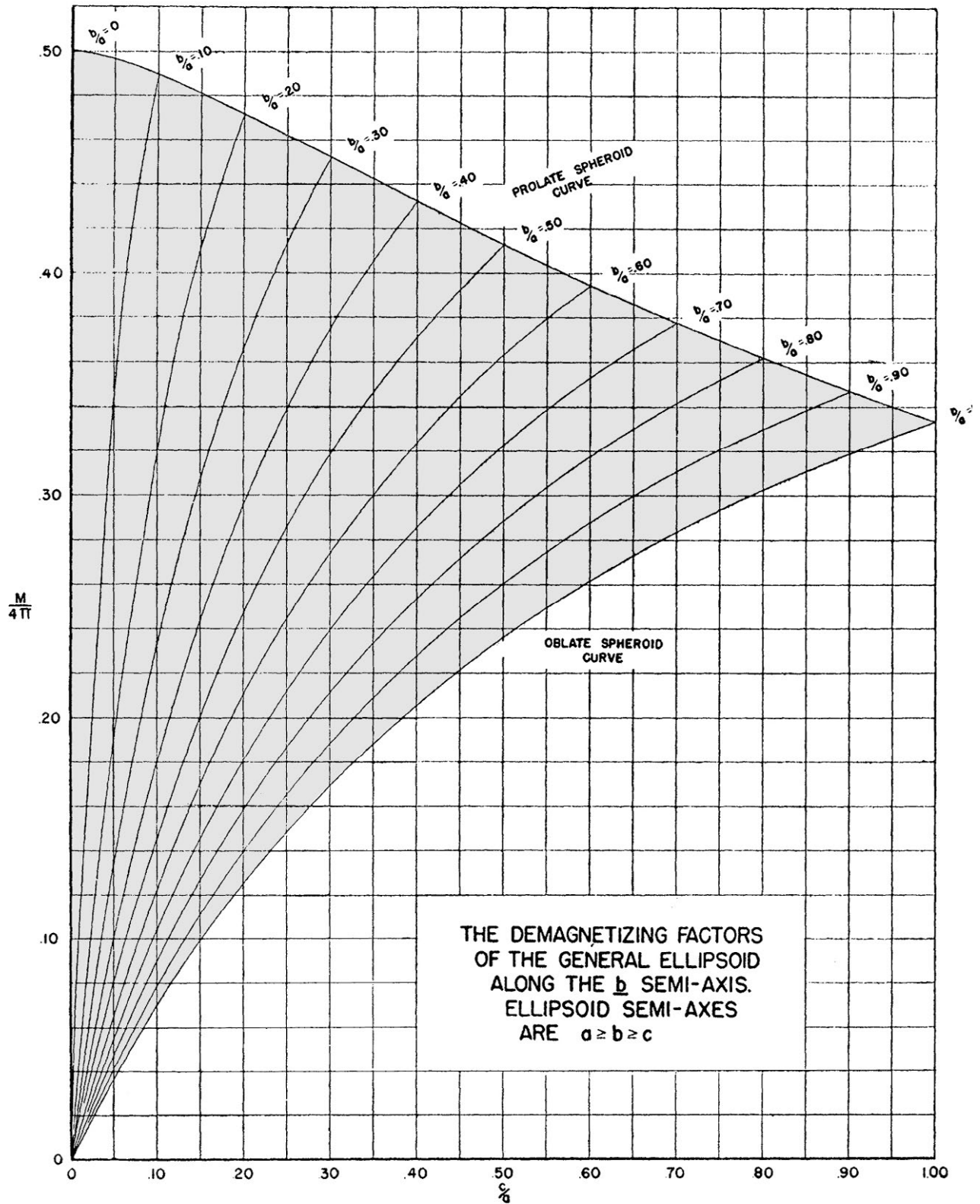


Figure 15. N_b ($M/4\pi$) as a function of c/a (γ), where each curve corresponds to a different value of the parameter b/a (β) (taken from Osborn, Figure 2).

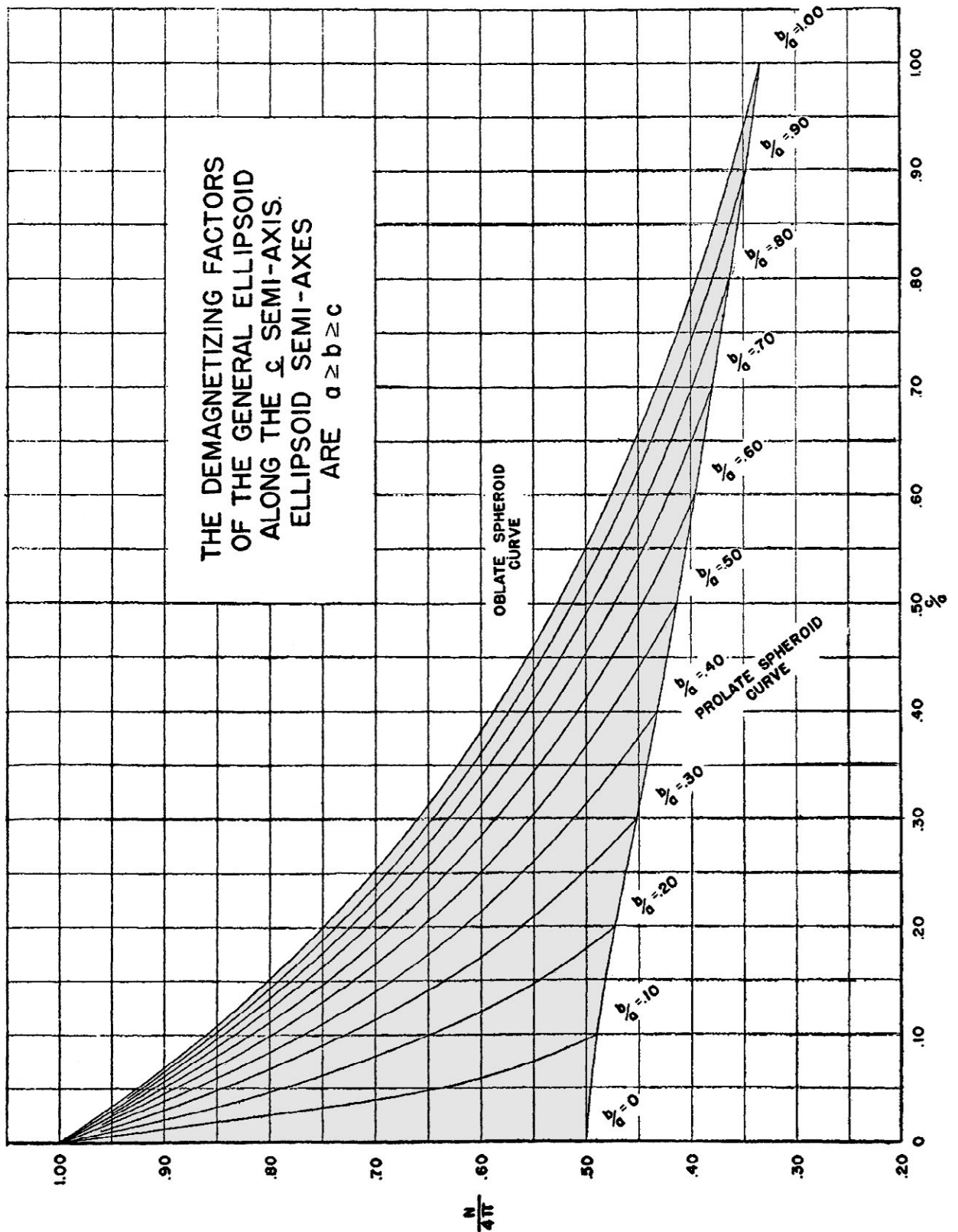


Figure 16. N_c ($N/4\pi$) as a function of c/a (γ), where each curve corresponds to a different value of the parameter b/a (β) (taken from Osborn, Figure 3).

Therefore

$$1/3 \leq N_c(1, \gamma_0, \gamma_0) \leq N_c(1, \beta, \gamma_0) \leq N_c(1, 1, \gamma_0) \leq 1. \quad (4.49)$$

For fixed γ_0 , as shown by the constant β curves, N_c is an increasing function of β ; that is

$$\text{if } \beta_2 > \beta_1, N_a(\beta_2, \gamma_0) > N_a(\beta_1, \gamma_0). \quad (4.50)$$

Each of the curves drawn correspond to a different value of the parameter $\beta = b/a$, the highest being the one for the oblate spheroid. The right vertex $N_c(1, 1) = N_b(1, 1) = N_c(1, 1) = 1/3$ corresponds to the sphere, the common limit of the oblate and prolate spheroids. The lowest left vertex, $N_c(0, 0) = N_b(0, 0) = 1/2$, characterizes the right elliptic cylinder of infinite length; the highest left vertex, $N_c(1, 0) = 1$, the constant thickness sheet of infinite extension (see Table 3).

Combined graph

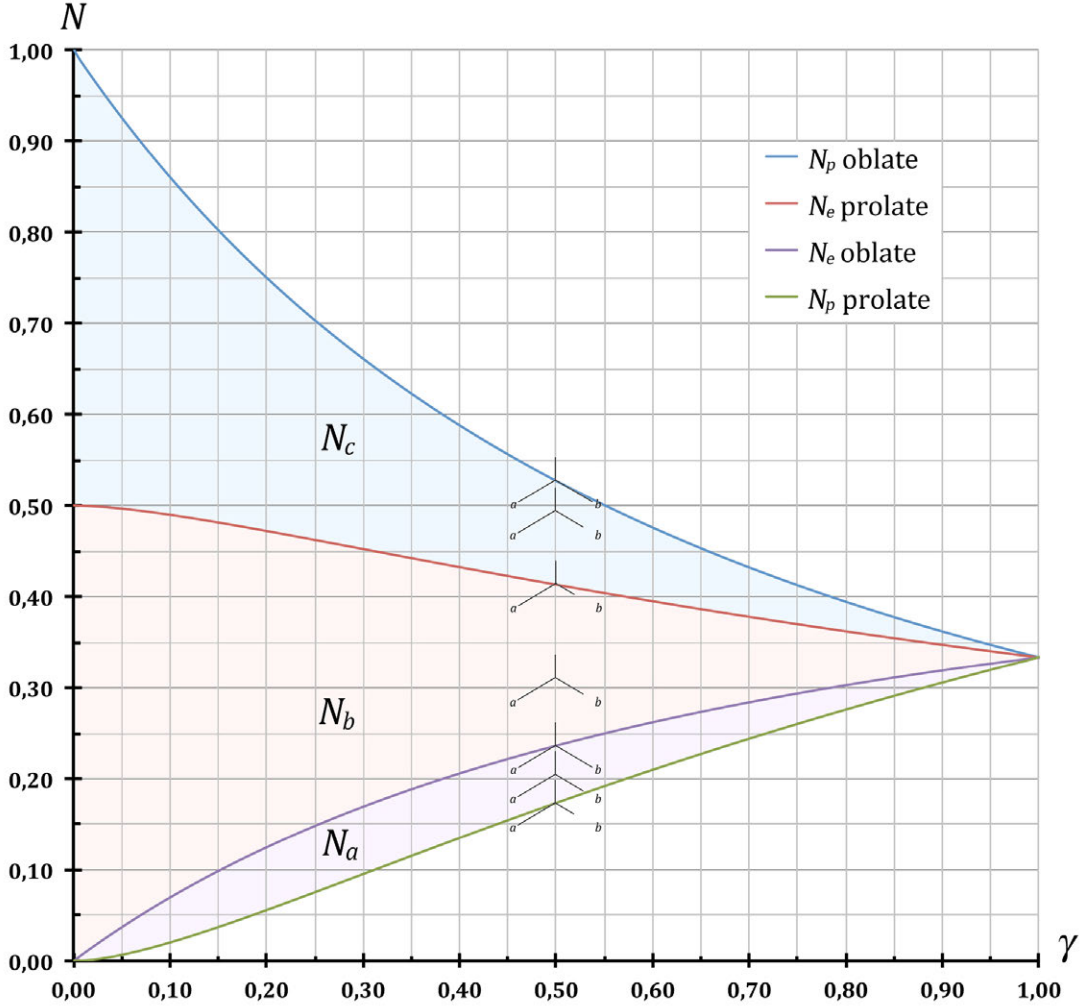


Figure 17. Distribution of eigenvalues of N .

In Figure 17 the graphs for the three eigenvalues are piled up, identifying with different colors the regions and boundary curves for each one¹²¹, where

¹²¹ A similar one was given in Stoner, Figure 1.

eigenvalues N_p correspond to the polar axis and N_e to the equatorial ones of the spheroids. As an example, the segments of allowed values for each are shown for $\gamma = 0.25$, illustrating the ordering given by eq. 3.72. The constant β curves have been omitted for simplicity.

The reader should exercise its understanding of the graphs with activities like the one proposed in Problem 22 and incorporated into Figure 17. The ellipsoids there analyzed are represented as seen from a direction at equal angles with the three semiaxes, and the length of each (c is the vertical one) is represented as the corresponding fraction of a . The value of γ is kept constant ($\frac{1}{2}$) and that of β is $\frac{1}{2}$, $\frac{3}{4}$ or 1. For N_a , on the curve for the polar eigenvalue of the prolate spheroid, b is equal to c ($N_b = N_c$). When going up along the line $\gamma = \frac{1}{2}$ the b semiaxis increases for N_a until it is equal to a on the curve for the equatorial eigenvalue of the oblate spheroid ($N_a = N_b$). For N_b the b semiaxis decreases until it is equal to c ($N_b = N_c$) on the curve for the equatorial eigenvalue of the prolate spheroid. The behaviour is again reversed for N_c , where b increases until it is again equal to a ($N_a = N_b$) on the curve for the polar eigenvalue of the oblate spheroid.

The range of the different eigenvalues is

$$0 \leq N_a \leq 1/3, 0 \leq N_b \leq 1/2, 1/3 \leq N_c \leq 1, \quad (4.51)$$

so that N_c is the only eigenvalue that may be univocally identified from its value in the range $[1/2, 1]$ (see problem Problem 20).

External depolarization tensor

Obtention of the external gravitational potential by Ivory's method

In the previous sections the value of the gravitational potential $V(\vec{r})$ inside a solid homogeneous general ellipsoid, eq. 4.2, was used to derive the values of the internal depolarization tensor \mathbf{N} . If a similar procedure were used for obtaining the value outside the body, \mathbf{n}^{ext} , the expressions obtained would be extremely cumbersome. More than a century ago, in 1809, the Scottish mathematician James Ivory¹²² gave a concise way of deriving the potential's external values from the internal ones using ellipsoids confocal with the body's surface eq. A7.2. These confocal ellipsoids are obtained from eq. A7.20, where $\kappa > 0$ is usually defined as the largest algebraic root of that equation¹²³, and its value is the same for all \vec{r} on that surface¹²⁴. These confocal ellipsoidal surfaces are larger than the body, enclose it completely and have the same potential as certain corresponding points inside the body (see section *Confocal ellipsoids* of Appendix 7). Only the final expressions obtained using Ivory's method are given here. Readers interested in

¹²² James Ivory at English Wikipedia.

¹²³ But see eq. Problem 20.

¹²⁴ For a better understanding of the method the reader should start by reading section *Confocal ellipsoids* and tackling afterwards Problem 32, its application for the obtention of the external potential of a solid and homogeneous spherical mass.

learning how to obtain them may look at the relevant sections of Macmillan's book¹²⁵.

The potential V and the auxiliary function f eq. 3.8 are then given by¹²⁶,

$$f(\vec{r}) = -\frac{V(\vec{r})}{4\pi\sigma} = \frac{1}{2} \left(-N_0(\kappa) + N_a(\kappa)x^2 + N_b(\kappa)y^2 + N_c(\kappa)z^2 \right), \quad (4.52)$$

where the coefficients $N_\alpha(0)$ ($\alpha = x, y, z$ or a, b, c) are the eigenvalues of the internal depolarization tensor, but also appear in the expression of the external one for $\kappa \neq 0$. These coefficients are given by the following integrals

$$N_0(\kappa) = \frac{abc}{2} \int_{\kappa(\vec{r})}^{\infty} \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \quad (4.53)$$

$$N_\alpha(\kappa) = \frac{abc}{2} \int_{\kappa(\vec{r})}^{\infty} \frac{ds}{\sqrt{(d_\alpha^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}},$$

where $d_x = a, d_y = b, d_z = c$, $\alpha = x, y, z$ and

$$\frac{x^2}{a^2+\kappa} + \frac{y^2}{b^2+\kappa} + \frac{z^2}{c^2+\kappa} = 1. \quad (4.54)$$

The expressions of the integrals in terms of Legendre's elliptic functions are given at the end of [Appendix 9](#)¹²⁷, where more details are given about these functions. The corresponding values for $N_j(\kappa)$ are

$$N_0(\kappa) = \frac{abc}{\sqrt{a^2-c^2}} F(\phi(\kappa), k), \quad (4.55)$$

$$N_a(\kappa) = \frac{abc}{\sqrt{a^2-c^2}(a^2-b^2)} \left(-E(\phi(\kappa), k) + F(\phi(\kappa), k) \right),$$

(4.56)

¹²⁵ MacMillan, sections 35-36, pp. 52-58.

¹²⁶ MacMillan, p. 56, eq. 36.1.

¹²⁷ See also Bartczak & Breiter & Jusiel; [Ellipsoids, material points and material segments](#); Celestial Mech. Dyn. Astr.; eq. 6.

$$\begin{aligned}
N_b(\kappa) &= -\frac{abc\sqrt{c^2+\kappa}}{\sqrt{(a^2+\kappa)(b^2+\kappa)(b^2-c^2)}} + \frac{abc\sqrt{a^2-c^2}}{(a^2-b^2)(b^2-c^2)}E(\phi(\kappa),k) \\
&\quad - \frac{abc}{\sqrt{a^2-c^2}(a^2-b^2)}F(\phi(\kappa),k), \\
N_c(\kappa) &= \frac{abc\sqrt{b^2+\kappa}}{\sqrt{(a^2+\kappa)(c^2+\kappa)(b^2-c^2)}} - \frac{abc}{\sqrt{(a^2-c^2)(b^2-c^2)}}E(\phi(\kappa),k), \\
k &= \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \frac{x^2}{a^2+\kappa} + \frac{y^2}{b^2+\kappa} + \frac{z^2}{c^2+\kappa} = 1, \quad \phi(\kappa) = \arcsin\left(\sqrt{\frac{a^2-c^2}{a^2+\kappa}}\right).
\end{aligned} \tag{4.56}$$

Notice that the only difference with the interior values of the potential eq. 4.4 is that the lower limit 0 of the integrals is here replaced by $\kappa(\vec{r})$, origin of the notation $N_\alpha(0)$.

Near and far away point approximations to n^{ext}

Expressions for the near or far away points are the simplest approximations to the external field of a uniformly polarized ellipsoid body, both when the polarization is permanent or induced by an applied field.

For field points very near to the body's surface the values eqs. 3.41 and 3.42 may be used, coinciding with eq. 4.66 for $\kappa = 0$:

$$\mathbf{n}^{\text{ext}}(\vec{r}^S) = \mathbf{N}(0) - \begin{pmatrix} \frac{x^2/a^4}{w(\vec{r})} & \frac{x \cdot y/a^2b^2}{w(\vec{r})} & \frac{x \cdot z/a^2c^2}{w(\vec{r})} \\ \frac{x \cdot y/a^2b^2}{w(\vec{r})} & \frac{y^2/b^4}{w(\vec{r})} & \frac{y \cdot z/b^2c^2}{w(\vec{r})} \\ \frac{x \cdot z/a^2c^2}{w(\vec{r})} & \frac{y \cdot z/b^2c^2}{w(\vec{r})} & \frac{z^2/c^4}{w(\vec{r})} \end{pmatrix}, \tag{4.57}$$

where $w(\vec{r}) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$.

For points far away from the uniformly polarized body it is expected that the field will be the dipolar one (the extension of the second of eqs. 3.66) generated by the dipole moment of the body, presumption that is confirmed by a calculation of the limit value of κ and the resulting external tensor (see Problem 33).

General expression of external n

From eq. 3.4 the components of the depolarization tensor, either inside or outside the ellipsoidal body, are given by

$$n_{\alpha\beta}^{\text{ext}}(\vec{r}) = \frac{\partial^2 f(\vec{r})}{\partial x_\alpha \partial x_\beta}, \quad \text{where } \vec{r} \notin V. \quad (4.58)$$

For the purpose of taking first derivatives it is convenient to express f , eqs. 4.52 and 4.54, as the following single integral:

$$f(\vec{r}) = \frac{1}{2} \int_{\kappa(\vec{r})}^{\infty} \left(\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} - 1 \right) ds = \frac{1}{2} \int_{\kappa(\vec{r})}^{\infty} Q(s, \vec{r}) ds. \quad (4.59)$$

Written in this fashion the integrand $Q(s, \vec{r})$ vanishes for the upper and lower limit of the integral (see eq. A7.20), so that

$$\begin{aligned} \frac{\partial}{\partial x_\beta} \int_{\kappa(\vec{r})}^{\infty} Q(s, \vec{r}) ds &= \frac{\partial}{\partial \kappa} \left(\int_{\kappa(\vec{r})}^{\infty} Q(s, \vec{r}) ds \right) \frac{\partial \kappa}{\partial x_\beta} + \int_{\kappa(\vec{r})}^{\infty} \frac{\partial}{\partial x_\beta} Q(s, \vec{r}) ds \\ &= -Q(s, \vec{r}) \Big|_{\kappa(\vec{r})}^{\infty} \frac{\partial \kappa}{\partial x_\beta} + \int_{\kappa(\vec{r})}^{\infty} \frac{\partial}{\partial x_\beta} Q(s, \vec{r}) ds = \int_{\kappa(\vec{r})}^{\infty} \frac{\partial}{\partial x_\beta} Q(s, \vec{r}) ds \\ &= \frac{abc}{2} \int_{\kappa(\vec{r})}^{\infty} \frac{2x_\beta}{(d^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}} ds = N_\beta(\kappa) x_\beta. \end{aligned} \quad (4.60)$$

where eq. 4.54 has been used.

That is,

$$\frac{\partial f(\vec{r})}{\partial x_\beta} = N_\beta(\kappa(\vec{r})) x_\beta, \quad (4.61)$$

where $N_0(\kappa)$ does not appear.

The calculation of the second derivatives of f gives

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial f(\vec{r})}{\partial x_\beta} \right) &= \frac{\partial}{\partial x_\alpha} (N_\beta(\kappa(\vec{r})) x_\beta) \\ &= N_\beta(\kappa(\vec{r})) \delta_{\alpha\beta} + x_\beta \frac{\partial}{\partial \kappa} (N_\beta(\kappa)) \frac{\partial \kappa(\vec{r})}{\partial x_\alpha}, \end{aligned} \quad (4.62)$$

requires the value of $\partial \kappa / \partial x_\beta$, which may be obtained from its implicit definition eq. A7.20. The result will shed light on the meaning of the external tensor and verify the body's surface value of \mathbf{n}^{ext} eq. 3.41.

From eq. A7.20,

$$\begin{aligned}
 \frac{\partial}{\partial x_\beta} \left(\frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} \right) &= \frac{\partial}{\partial x_\beta} \sum_{\alpha=x,y,z} x_\alpha^2 (d_\alpha^2 + \kappa)^{-1} \\
 &= \frac{2x_\beta}{d_\beta^2 + \kappa} + \frac{\partial}{\partial \kappa} \left(\sum_{\alpha} x_\alpha^2 (d_\alpha^2 + \kappa)^{-1} \right) \frac{\partial \kappa}{\partial x_\beta} \\
 &= \frac{2x_\beta}{d_\beta^2 + \kappa} - \left(\sum_{\alpha} x_\alpha^2 (d_\alpha^2 + \kappa)^{-2} \right) \frac{\partial \kappa}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} 1 = 0.
 \end{aligned} \tag{4.63}$$

That is,

$$\frac{\partial \kappa}{\partial x_\beta} = \frac{\frac{2x_\beta}{d_\beta^2 + \kappa}}{\left(\frac{x^2}{(a^2 + \kappa)^2} + \frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2} \right)}, \tag{4.64}$$

which, as may be seen from eq. A7.8, defines the components of a vector normal to the surface of the confocal ellipsoid A7.20.

The derivatives of N_β give

$$\begin{aligned}
 \frac{\partial N_\beta(\kappa)}{\partial x_\alpha} &= \frac{\partial N_\beta(\kappa)}{\partial \kappa} \frac{\partial \kappa}{\partial x_\beta} \\
 &= \frac{\partial}{\partial \kappa} \left(\frac{abc}{2} \int_{\kappa(\vec{r})}^{\infty} \frac{ds}{(d_\beta + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \right) \frac{\partial \kappa}{\partial x_\beta} \\
 &= -abc \frac{\left(\frac{x_\alpha}{d_\alpha^2 + \kappa} \right) / \left(\frac{x^2}{(a^2 + \kappa)^2} + \frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2} \right)}{(d_\beta + \kappa) \sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}}.
 \end{aligned} \tag{4.65}$$

From eqs. 4.61, 4.62 and 4.65 it is thus obtained

$$\begin{aligned}
 n_{\alpha\beta}^{\text{ext}}(\vec{r}) &= \frac{\partial^2 f(\vec{r})}{\partial x_\alpha \partial x_\beta} = N_\beta(\kappa) \delta_{\alpha\beta} + \frac{\partial N_\beta(\kappa)}{\partial x_\alpha} x_\beta \\
 &= N_\beta(\kappa) \delta_{\alpha\beta} - abc \frac{s_\alpha(\vec{r}, \kappa) s_\beta(\vec{r}, \kappa)}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}}.
 \end{aligned}$$

Therefore¹²⁸

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \mathbf{N}(\kappa) - abc \frac{\hat{s}(\vec{r}|\kappa) \hat{s}(\vec{r}|\kappa)}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}},$$

where $N_{\alpha\beta}(\kappa) = \delta_{\alpha\beta} \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(d_{\alpha}^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}}$ (4.66)

and $\hat{s}(\vec{r}|\kappa)$ is the unit vector normal to the surface of the κ confocal ellipsoid eq. A7.20 at the external field point \vec{r} :

$$s_{\alpha}(\vec{r}|\kappa) = \frac{\frac{x_{\alpha}}{d_{\alpha}^2 + \kappa}}{\sqrt{\frac{x^2}{(a^2 + \kappa)^2} + \frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2}}},$$

where $\frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1$. (4.67)

Notice that the first term of \mathbf{n}^{ext} is a field point dependent extension of the internal depolarization tensor \mathbf{N} and proportional to its volume. In the second term the numerator selects components of the polarization normal to the confocal ellipsoid, while the denominator gives a distance of the order of r^3 (see eq. A7.22). For $\kappa = 0$ it is obtained the value of \mathbf{n}^{ext} at the surface of the body eq. 3.41.

Traces

In what follows it is verified if eq. 4.66 fulfills the zero trace rule eqs. 3.15.

$$\text{Tr } \mathbf{n}^{\text{ext}}(\vec{r}) = \sum_{\alpha} n_{\alpha\alpha}^{\text{ext}}(\vec{r}) = \sum_{\alpha} N_{\alpha\alpha}(\kappa) - abc \frac{\left(\sum_{\alpha} s_{\alpha}^2(\vec{r}|\kappa) \right) / |s(\vec{r}|\kappa)|^2}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}}. \quad (4.68)$$

¹²⁸ Barczak & t & Jusiel; *Ellipsoids, material points and material segments*; Celestial Mech. Dyn. Astr. ; DOI 10.1007/s10569-006-9017-x. The value coincides with the Hessian of the gravitational potential V , eqs. 9 and 10.

From the definitions of N_α eqs. 4.54 it follows that

$$\begin{aligned}
 \sum_{\alpha} N_{\alpha\alpha}(\kappa) &= \frac{a \cdot b \cdot c}{2} \int_{\kappa}^{\infty} \left(\frac{1}{a^2+s} + \frac{1}{b^2+s} + \frac{1}{c^2+s} \right) \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} \\
 &= \frac{abc}{2} \int_{\kappa}^{\infty} \frac{(b^2+s)(c^2+s) + (a^2+s)(c^2+s) + (a^2+s)(b^2+s)}{\left[(a^2+s)(b^2+s)(c^2+s) \right]^{3/2}} ds \\
 &= \frac{abc}{2} \int_{\kappa}^{\infty} \frac{d}{ds} \left(\frac{(a^2+s)(b^2+s)(c^2+s)}{\left[(a^2+s)(b^2+s)(c^2+s) \right]^{3/2}} \right) ds = - \frac{abc}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} \Bigg|_{\kappa}^{\infty} \\
 &= \frac{abc}{\sqrt{(a^2+\kappa)(b^2+\kappa)(c^2+\kappa)}}.
 \end{aligned} \tag{4.69}$$

That is

$$\text{Tr } \mathbf{N}(\kappa) = \frac{abc}{\sqrt{(a^2+\kappa)(b^2+\kappa)(c^2+\kappa)}}, \tag{4.70}$$

which, as its eigenvalues $N_a(\kappa)$, $N_b(\kappa)$ and $N_c(\kappa)$, is of the order of r^3 (see eq. A7.22) and may be used to simplify its calculation.

From eq. A7.18,

$$\frac{\sum_{\alpha} s_{\alpha}^2(\vec{r}|\kappa)}{|s(\vec{r}|\kappa)|^2} = 1. \tag{4.71}$$

Upon replacement in the initial expression for the trace, it is verified that

$$\text{Tr } \mathbf{n}^{\text{ext}}(\vec{r}) = 0. \tag{4.72}$$

Verification for the sphere

Equations 4.66 and 4.67 will next be verified using the solution found for the sphere using the electrostatic Gauss's Law eqs. 3.66 and 3.67. To that end the different components of those equations are evaluated next.

The eigenvalues of $\mathbf{N}(\kappa)$ are

$$\begin{aligned}
 N_a(\kappa) = N_b(\kappa) = N_c(\kappa) &= \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(d_{\alpha}^2+s) \sqrt{(a^2+s)(b^2+s)(c^2+s)}} \\
 &= \frac{R^3}{2} \int_{\kappa}^{\infty} \frac{ds}{(R^2+s)^{5/2}} = - \frac{R^3}{3} (R^2+s)^{-3/2} \Bigg|_{\kappa}^{\infty} = \frac{R^3}{3} (R^2+\kappa)^{-3/2},
 \end{aligned} \tag{4.73}$$

where $a = b = c = R$. The value of κ to be used is the solution of the equation

$$\frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = \frac{r^2}{R^2 + \kappa} = 1, \quad \kappa = r^2 - R^2. \quad (4.74)$$

Therefore

$$N_a(\kappa) = N_b(\kappa) = N_c(\kappa) = \frac{R^3}{3(R^2 + \kappa)^{3/2}} = \frac{1}{3} \frac{R^3}{r^3}, \quad (4.75)$$

its value being $1/3$ for the sphere's interior where $\kappa = 0$.

The last term of eq. 4.66 has in the the numerator

$$\hat{s}(\vec{r} | \kappa) = \sum_{\alpha} s_{\alpha}(\vec{r} | \kappa) \hat{x}_{\alpha} = \sum_{\alpha} \frac{x_{\alpha} \hat{x}_{\alpha}}{r} = \frac{\vec{r}}{r}, \quad (4.76)$$

and the denominator is

$$\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)} = r^3, \quad (4.77)$$

giving

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \frac{1}{3} \frac{R^3}{r^3} \mathbf{1} - R^3 \frac{\vec{r}\vec{r}}{r^5} = \frac{1}{3} R^3 \left(\frac{1}{r^3} \mathbf{1} - \frac{3\vec{r}\vec{r}}{r^5} \right). \quad (4.78)$$

This is the dyadic such that both for permanent or induced electric (eqs. 2.7 and 2.32a) and magnetic polarization (eqs. 2.40 and 2.48) gives the following external dipolar fields contributions (see eqs. 3.66 and A4.1):

$$\begin{aligned} \vec{E}_{\vec{p}}(\vec{r}) &= k_1 \frac{3(\vec{p} \cdot \vec{r})\vec{r} - r^2 \vec{p}}{r^5}, \\ \vec{H}_{\vec{m}}(\vec{r}) &= \frac{\lambda'}{4\pi} \frac{3(\vec{m} \cdot \vec{r})\vec{r} - r^2 \vec{m}}{r^5}, \end{aligned} \quad (4.79)$$

where eqs. A1.4 and A1.11 have been used.

Elliptic cylinder

The first term of eq. 4.66 may be obtained from eqs. 4.12, 4.13 and 4.14¹²⁹:

$$\begin{aligned} N_a^{\text{ec}}(\kappa) &= 0, \\ N_b^{\text{ec}}(\kappa) &= \frac{bc}{b^2 - c^2} \left(\frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{b^2 + \kappa}} \right), \end{aligned} \quad (4.80)$$

¹²⁹ These expressions coincide with those derived from the potential given by MacMillan at page 71.

$$N_c^{\text{ec}}(\kappa) = \frac{bc}{b^2 - c^2} \left(\frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{c^2 + \kappa}} \right). \quad (4.80)$$

Taking limit for $a = \infty$ in the second term of eq. 4.66 gives

$$\begin{aligned} & \lim_{a \rightarrow \infty} \frac{abc}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}} \\ &= \lim_{a \rightarrow \infty} \frac{bc}{\sqrt{\left(1 + \frac{\kappa}{a^2}\right)(b^2 + \kappa)(c^2 + \kappa)}} = \frac{bc}{\sqrt{(b^2 + \kappa)(c^2 + \kappa)}}, \end{aligned} \quad (4.81)$$

$$s_y(\vec{r}|\kappa) = \frac{\frac{y}{b^2 + \kappa}}{\sqrt{\frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2}}}, \quad s_z(\vec{r}|\kappa) = \frac{\frac{z}{c^2 + \kappa}}{\sqrt{\frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2}}}. \quad (4.82)$$

As coordinate x is absent, the value of κ is given here by (see eq. A7.23)

$$\frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1 \quad (4.83)$$

From eq. 4.70,

$$\begin{aligned} \text{Tr } \mathbf{N}^{\text{ec}}(\kappa) &= N_a^{\text{ec}}(\kappa) + N_b^{\text{ec}}(\kappa) + N_c^{\text{ec}}(\kappa) \\ &= 0 + \frac{bc}{b^2 - c^2} \left(\frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{b^2 + \kappa}} + \frac{\sqrt{b^2 + \kappa} - \sqrt{c^2 + \kappa}}{\sqrt{c^2 + \kappa}} \right) \\ &= \frac{bc}{\sqrt{(b^2 + \kappa)(c^2 + \kappa)}} = \lim_{a \rightarrow \infty} \frac{abc}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}}. \end{aligned} \quad (4.84)$$

For $b = c$ the right circular cylinder is obtained. The value of κ is obtained from

$$\frac{y^2}{b^2 + \kappa} + \frac{z^2}{b^2 + \kappa} = 1, \quad \text{so that } \kappa = r^2 - b^2. \quad (4.85)$$

The eigenvalues of \mathbf{N} should be recalculated from eqs. 4.13 and 4.14,

$$\begin{aligned} N_b^{\text{ec}}(\kappa) &= \frac{bc}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2 + s)^{3/2} \sqrt{c^2 + s}} = \frac{b^2}{2} \int_{\kappa}^{\infty} \frac{ds}{(b^2 + s)^2} \\ &= -\frac{1}{2} \frac{b^2}{b^2 + s} \Big|_{\kappa}^{\infty} = \frac{1}{2} \frac{b^2}{b^2 + \kappa} = \frac{1}{2} \frac{b^2}{r^2}, \end{aligned} \quad (4.86)$$

where eq. 4.85 was used.

The coefficient and the factors of matrix **S** are

$$\frac{b^2}{b^2+\kappa} = \frac{b^2}{r^2}, \quad s_\alpha(\vec{r}|\kappa) = \frac{\frac{x_\alpha}{b^2+\kappa}}{\sqrt{\frac{y^2+z^2}{(b^2+\kappa)^2}}} = \frac{x_\alpha}{r}, \quad (4.87)$$

so that for the right circular cylinder of infinite length

$$\begin{aligned} \mathbf{n}^{\text{ext}}(\vec{r}) &= \mathbf{N}(\kappa) - \frac{bc}{\sqrt{(b^2+\kappa)(c^2+\kappa)}} \mathbf{S}(\vec{r}|\kappa) \\ &= \frac{b^2}{r^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} - \frac{b^2}{r^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^2/r^2 & yz/r^2 \\ 0 & yz/r^2 & z^2/r^2 \end{pmatrix} \\ &= \frac{b^2}{r^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 - y^2/r^2 & -yz/r^2 \\ 0 & -yz/r^2 & 1/2 - z^2/r^2 \end{pmatrix}. \end{aligned} \quad (4.88)$$

the very same zero trace eq. 3.58 obtained using Gauss's Law.

Summarizing, for the right elliptic cylinder of infinite extension:

$$\begin{aligned} \mathbf{n}^{\text{ext}}(\vec{r}) &= \mathbf{N}(\kappa) - \frac{bc}{\sqrt{(b^2+\kappa)(c^2+\kappa)}} \mathbf{S}(\vec{r}, \kappa), \quad \mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_b^{\text{ec}} & 0 \\ 0 & 0 & N_c^{\text{ec}} \end{pmatrix}, \\ N_b^{\text{ec}}(\kappa) &= \frac{bc}{b^2-c^2} \left(\frac{\sqrt{b^2+\kappa} - \sqrt{c^2+\kappa}}{\sqrt{b^2+\kappa}} \right), \quad N_c^{\text{ec}}(\kappa) = \frac{bc}{b^2-c^2} \left(\frac{\sqrt{b^2+\kappa} - \sqrt{c^2+\kappa}}{\sqrt{c^2+\kappa}} \right), \\ \mathbf{S}(\vec{r}, \kappa) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_y^2 & s_y s_z \\ 0 & s_y s_z & s_z^2 \end{pmatrix}, \quad s_\alpha(\vec{r}, \lambda) = \frac{\frac{x_\alpha}{d^2+\kappa}}{\sqrt{\frac{y^2}{(b^2+\kappa)^2} + \frac{z^2}{(c^2+\kappa)^2}}}, \\ \frac{y^2}{b^2+\kappa} + \frac{z^2}{c^2+\kappa} &= 1, \quad \alpha = y, z, \quad d_y = b, d_z = c. \end{aligned} \quad (4.89)$$

Oblate spheroid

For oblate spheroids $a = b$, eq. 4.66 gives¹³⁰

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \mathbf{N}(\kappa) - a^2 c \frac{\hat{s}(\vec{r}, \kappa) \hat{s}(\vec{r}, \kappa)}{(a^2 + \kappa) \sqrt{(c^2 + \kappa)}}, \quad (4.90)$$

where

$$\frac{x^2 + y^2}{a^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1, \quad (4.91)$$

$$\mathbf{N}(\kappa) = \begin{pmatrix} N_e(\kappa) & 0 & 0 \\ 0 & N_e(\kappa) & 0 \\ 0 & 0 & N_p(\kappa) \end{pmatrix}. \quad (4.92)$$

From eqs. 4.20 and 4.26,

$$\begin{aligned} N_e^o(\kappa) &= N_a^o(\kappa) = N_b^o(\kappa) \\ &= -\frac{1}{2} \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{1}{2} \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan \left(\frac{\sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}} \right). \end{aligned} \quad (4.93)$$

$$\begin{aligned} N_p^o(\kappa) &= N_c^o(\kappa) \\ &= \frac{a^2 c}{(a^2 - c^2) \sqrt{c^2 + \kappa}} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan \left(\frac{\sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}} \right). \end{aligned} \quad (4.94)$$

From eqs. and 4.67 and A7.27,

$$\hat{s}_\alpha(\vec{r}, \kappa) = \frac{\frac{x\hat{x} + y\hat{y}}{a^2 + \kappa} + \frac{z\hat{z}}{c^2 + \kappa}}{\sqrt{\frac{x^2 + y^2}{(a^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2}}}, \quad \frac{(x^2 + y^2)}{a^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1. \quad (4.95)$$

Eqs. 4.20 and 4.26 reduce to the formulas for the infinite sheet (see eq. 3.47) for $a = b \rightarrow \infty$ and to those of the sphere (see eq. 3.66) for $a = b = c$.

The trace eq. 4.70 is satisfied because

¹³⁰ MacMillan, p. 62 eq. 39.2.

$$\begin{aligned}
\text{Tr } \mathbf{N}(\kappa) &= N_a^0(\kappa) + N_b^0(\kappa) + N_c^0(\kappa) = 2N_e^0(\kappa) + N_p^0(\kappa) \\
&= -\frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} + \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan\left(\frac{\sqrt{a^2 - c^2}}{\sqrt{c^2 + \kappa}}\right) \\
&\quad + \frac{a^2 c}{(a^2 - c^2)\sqrt{c^2 + \kappa}} - \frac{a^2 c}{(a^2 - c^2)^{3/2}} \arctan\sqrt{\frac{a^2 - c^2}{c^2 + \kappa}} \\
&= \frac{a^2 c}{(a^2 - c^2)\sqrt{c^2 + \kappa}} - \frac{a^2 c \sqrt{c^2 + \kappa}}{(a^2 - c^2)(a^2 + \kappa)} = \frac{a^2 c}{(a^2 + \kappa)\sqrt{c^2 + \kappa}} = \frac{aac}{\sqrt{(a^2 + \kappa)(a^2 + \kappa)(c^2 + \kappa)}}.
\end{aligned} \tag{4.96}$$

Prolate spheroid

For prolate spheroids $a > b = c$, eq. 4.66 gives¹³¹

$$\mathbf{n}^{\text{ext}}(\vec{r}) = \mathbf{N}^p(\kappa) - ac^2 \frac{\hat{s}(\vec{r}, \kappa) \hat{s}(\vec{r}, \kappa)}{\sqrt{(a^2 + \kappa)(c^2 + \kappa)}}, \tag{4.97}$$

where

$$\frac{x^2}{a^2 + \kappa} + \frac{y^2 + z^2}{c^2 + \kappa} = 1, \tag{4.98}$$

$$\mathbf{N}^p(\kappa) = \begin{pmatrix} N_p^p(\kappa) & 0 & 0 \\ 0 & N_e^p(\kappa) & 0 \\ 0 & 0 & N_e^p(\kappa) \end{pmatrix}. \tag{4.99}$$

From eqs. 4.31 and 4.37,

$$\begin{aligned}
N_p^p(\kappa) &= N_a^p(\kappa) \\
&= -\frac{ac^2}{(a^2 - c^2)\sqrt{a^2 + \kappa}} + \frac{1}{2} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{a^2 + \kappa} - \sqrt{a^2 - c^2}} \right),
\end{aligned} \tag{4.100}$$

$$\begin{aligned}
N_e^p(\kappa) &= N_b^p(\kappa) = N_c^p(\kappa) \\
&= \frac{1}{2} \frac{ac^2 \sqrt{a^2 + \kappa}}{(a^2 - c^2)(c^2 + \kappa)} - \frac{1}{4} \frac{ac^2}{(a^2 - c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2 + \kappa} + \sqrt{a^2 - c^2}}{\sqrt{a^2 + \kappa} - \sqrt{a^2 - c^2}} \right).
\end{aligned} \tag{4.101}$$

¹³¹ MacMillan, p. 62 eq. 39.2.

Finally, from eqs. and 4.67 and A7.27,

$$\hat{s}_\alpha(\vec{r}, \kappa) = \frac{\frac{x\hat{x}}{a^2+\kappa} + \frac{y\hat{y} + z\hat{z}}{c^2+\kappa}}{\sqrt{\frac{x^2}{(a^2+\kappa)^2} + \frac{y^2+z^2}{(c^2+\kappa)^2}}}, \quad \text{where} \quad \frac{x^2}{a^2+\kappa} + \frac{y^2+z^2}{c^2+\kappa} = 1. \quad (4.102)$$

The trace eq. 4.70 is satisfied because

$$\begin{aligned} \text{Tr } \mathbf{N}(\kappa) &= N_a^p(\kappa) + N_b^p(\kappa) + N_c^p(\kappa) = N_p^p(\kappa + 2N_e^p(\kappa)) \\ &= \frac{ac^2\sqrt{a^2+\kappa}}{(a^2-c^2)(c^2+\kappa)} - \frac{1}{2} \frac{ac^2}{(a^2-c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2+\kappa} + \sqrt{a^2-c^2}}{\sqrt{a^2+\kappa} - \sqrt{a^2-c^2}} \right) \\ &\quad - \frac{ac^2}{(a^2-c^2)\sqrt{a^2+\kappa}} + \frac{1}{2} \frac{ac^2}{(a^2-c^2)^{3/2}} \ln \left(\frac{\sqrt{a^2+\kappa} + \sqrt{a^2-c^2}}{\sqrt{a^2+\kappa} - \sqrt{a^2-c^2}} \right) \\ &= \frac{ac^2\sqrt{a^2+\kappa}}{(a^2-c^2)(c^2+\kappa)} - \frac{ac^2}{(a^2-c^2)\sqrt{a^2+\kappa}} = \frac{ac^2}{\sqrt{a^2+\kappa}(c^2+\kappa)} = \frac{acc}{\sqrt{(a^2+\kappa)(c^2+\kappa)(c^2+\kappa)}}. \end{aligned} \quad (4.103)$$

Eqs. 4.37 and 4.31 reduce to the formulas for the infinite circular cylinder for $a \rightarrow \infty$, $b = c$, and to those of the sphere for $a = b = c$.

Triaxial ellipsoid

From eqs. A9.3, A9.5, A9.6 and A9.7 the eigenvalues of $\mathbf{N}(\kappa)$ may be expressed in terms of Legendre's elliptic functions as follows.

$$\begin{aligned} N_a(\kappa) &= \frac{abc}{\sqrt{a^2-c^2}(a^2-b^2)} (-E(\phi, k) + F(\phi, k)), \\ N_b(\kappa) &= -\frac{abc\sqrt{c^2+\kappa}}{\sqrt{(a^2+\kappa)(b^2+\kappa)(b^2-c^2)}} \\ &\quad + \frac{abc\sqrt{a^2-c^2}}{(a^2-b^2)(b^2-c^2)} E(\phi, k) - \frac{abc}{\sqrt{a^2-c^2}(a^2-b^2)} F(\phi, k), \\ N_c(\kappa) &= \frac{abc\sqrt{b^2+\kappa}}{\sqrt{(a^2+\kappa)(c^2+\kappa)(b^2-c^2)}} - \frac{abc}{\sqrt{a^2-c^2}(b^2-c^2)} E(\phi, k), \\ \text{where } a \geq b \geq c, \quad k &= \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \sin \phi = \sqrt{\frac{a^2-c^2}{a^2+\kappa}}, \\ \frac{x^2}{a^2+\kappa} + \frac{y^2}{b^2+\kappa} + \frac{z^2}{c^2+\kappa} &= 1. \end{aligned} \quad (4.104)$$

Chapter 5: Energy, forces and cavities

Thermodynamics of electrostatic and magnetostatic energy

Basic concepts

Thermodynamics studies the macroscopic exchanges of energy (work, heat, internal and radiation energy...) between a material system and its surroundings, and the values taken by its state variables in the process. To that end state functions¹³² are defined such that —for reversible processes— its equilibrium values are a function only of the set of final values of the state variables and are independent of the path followed to obtain them¹³³. Introductory courses of thermodynamics discuss only the case of gases, where the state variables are volume V , pressure P and temperature T . The first two are clearly insufficient for elastic solids, and the electromagnetic variables have yet to be added.

The performance and interpretation of experiments with electric and magnetic fields and polarizations require a careful formulation of its thermodynamic behaviour that is seldom done in introductory texts on electromagnetism. As many difficulties are found in the way, a short but general overview will be given in this chapter in order to identify some important features of the problema and write down —without proof—the principal expressions required, identifying its source. The author is not familiar with any book giving a full analysis of the thermodynamics of electromagnetism, but there are several studies in international journals of physics¹³⁴.

The state variables required for the full thermodynamical characterization of elastic solids are listed below. They are grouped in pairs ("cause" and "effect"), through scalar and tensorial material's properties, by the differential relationship that connects them in the linear range:

- temperature T and entropy S , related by the heat capacity c : $dS = \frac{c}{T}dT$;
- mass m and gravitational field \vec{g} , related by the space metric;
- strain tensor \mathbf{e} and stress tensor $\boldsymbol{\sigma}$, related by the compliance tensor \mathbf{c} of elastic modulus: $d\boldsymbol{\sigma} = \mathbf{c} \bullet d\mathbf{e}$;

¹³² Landau and Lifschitz call them *thermodynamic potentials*, a name more evocative of its properties. See pp. 46-96.

¹³³ This characteristic is shared, for instance, by the electrostatic potential ϕ .

¹³⁴ See, for instance, *On Magnetic and Electrostatic Energy* and *The Thermodynamics of Magnetization* by E. A. Guggenheim, Proc. Roy. Soc. (London) vol. **A155**, pp. 49—101, 1936.

- electric field \vec{E} and electric displacement \vec{D} , related by the electric permeability tensor ϵ : $\vec{D} = \epsilon \cdot \vec{E}$;
- magnetic field \vec{H} and magnetic induction \vec{B} , related by the magnetic permeability tensor μ : $\vec{B} = \mu \cdot \vec{H}$.

For the magnetic case equilibrium refers to the steady state where electric currents are constant. It must also be stressed that the formulas given in this chapter are not valid for dynamic phenomena such as those involved in the interaction of electromagnetic radiation with matter, processes that are typically adiabatic.

Irreversible processes are characterized by an irreversible increase of entropy dS . If dQ is the quantity of heat transmitted in an elementary process, then

$$dQ = T \cdot dS. \quad (5.1)$$

Conduction currents in metals and magnetic hysteresis are typical electromagnetic irreversible (dissipative) processes. The atomic and molecular currents giving rise to the magnetization of solids are not dissipative, nor is the superconductor case. The theoretical calculation of electromagnetic energy computes only the reversible work required to establish a configuration of fields and polarizations in an ideal non dissipative process. In real processes —no matter how slow you make them— irreversible phenomena are always present. Thus, work dW done on a real system partially dissipates in internal (case of the electrical resistance) or external irreversible processes. Therefore

$$dW = dW_{\text{rev}} - T \cdot dS_{\text{irrev}}, \quad (5.2)$$

and the work computed in electromagnetic calculations is usually only the reversible part dW_{rev} .

The most important thermodynamic state function is the internal energy U that characterizes the balance between the quantity of heat dQ delivered to the system and the mechanical work dW done by the system on its surroundings,

$$dU = dQ - dW. \quad (5.3)$$

For solids the work expression used for gases, $dW = p \cdot dV$, must be replaced by

$$dW = \mathbf{c} \cdot d\mathbf{e}, \quad (5.4)$$

to which must be added the contributions of electromagnetic energy.

Newcomers to electromagnetism try to calculate internal energies by using forces to compute the external work required to build the system. This method, originated in mechanics, is reinforced in the study of gases, where pressures are introduced as surface distribution of forces. This is not the case in electromagnetism, where the starting point for calculating the variations of energy is not the concept of force or torque. On the contrary —as will be seen in this chapter— the latter are evaluated as the rate of change of electromagnetic energy respect to certain longitudes or angles.

An important part of the problem is that the variation of electromagnetic energy depends both on its spatial distribution and its time variation. If at this point the reader argues that we are considering here only static fields and steady state currents, some variations should be irrelevant, he should read again the introduction to eq. 5.2. The electromagnetic energy contained in any distribution of polarized matter is, by definition, that required to build that distribution from an initial state where polarizations are zero. This necessarily requires a time variation of polarizations and fields, process where induced currents make finite contributions to the energy of the system and generate radiation, however slowly the process is made. Moreover, fields are not limited to material bodies but extend throughout all space.

According to the type of process a material system undergoes, it is convenient to describe it using a peculiar state function different from U . This is done in a way such that its differential expression does not contain the differential of the state variable one wishes to keep constant. For instance, U is the more convenient state function for the description of processes where strains can be fixed, letting temperature T and stress σ to take the values that follow from the allowed heat exchange. Of course, any process may be described by any state function, but the theoretical analysis is simplified when using, *ceteris paribus*, an appropriate thermodynamic potential.

Thermodynamics of electromagnetism

The measurement of electromagnetic properties of polarized bodies is frequently, though not always, made in experiments where mechanical work is done. In those cases —the only discussed in this book—forces and torques are related to the variation of parameters characterizing the configuration of fields and polarizations. In a *ceteris paribus* fashion, several variables must be kept fixed throughout the process.

The body's intrinsic geometric configuration —the relative distribution of polarizable matter— is one of the variables that must be kept constant. This configuration is —in our case— the ellipsoidal shape and semiaxes that determine the fields's values, usually invariable when de , eq. 5.4, vanishes. As mentioned before, it is then convenient to use of the internal energy U . In practice, strain's control is not always possible, as happens in the phenomena of electrostriction¹³⁵, magnetostriction¹³⁶, piezoelectricity¹³⁷ and piezomagnetism, that require special treatment.

Electric and magnetic polarizations —dependent on the microscopic behaviour of atoms and molecules— are functions of temperature T , a variable that will be usually necessary to keep constant. As T is a “natural variable” of U , it is convenient to define the following thermodynamic potential

$$F = U - TS, \tag{5.5}$$

¹³⁵ Landau and Lifchitz, p. 55.

¹³⁶ Landau and Lifchitz, p. 155.

¹³⁷ Landau and Lifchitz, p. 73.

called Helmholtz free energy, for which

$$dF = T \cdot dS - dW - T \cdot dS - S \cdot dT = -dW - S \cdot dT. \quad (5.6)$$

The external work done in an isothermal process is then

$$dW = dW_{\text{rev}} - T \cdot dS_{\text{irrev.}} = -dF. \quad (5.7)$$

Any system tends to reorganize its elements in order to minimize its energy in the final state of equilibrium. When there are no external constraints, F varies until reaching a minimum dF vanishes and the system ceases to do external work. The reversible work delivered in the process of readjustment is

$$\Delta W_{\text{rev}} = \int dW_{\text{rev}} = -(\Delta F - T \cdot \Delta S_{\text{irrev.}}), \quad (5.8)$$

reason why F characterizes the maximum quantity of work that it is possible to extract from a system^{138,139}, concept that in chemistry is called affinity.

The internal energy U and the free energy F include an electromagnetic contribution. For F its value F_{em} is the amount of reversible work necessary to build up the final configuration of charges, electric currents and polarized matter. For the linear case (whose description requires several anisotropy tensors of rank two) its expression in terms of fields is¹⁴⁰

$$F_{\text{em}}|_{e=\text{const.}} = \frac{1}{2\lambda} \iiint_V \vec{E}(\vec{r}) \cdot \vec{D}(\vec{r}) d^3r + \frac{1}{2\lambda'} \iiint_V \vec{B}(\vec{r}) \cdot \vec{H}(\vec{r}) d^3r, \quad (5.9)$$

where the integral is taken over all the regions containing the sources of the fields, the ones where $\nabla \cdot \vec{D}$ and $\nabla \times \vec{H}$ are non-vanishing. There are no general expressions for the non-linear case, where each specific situation has to be dealt with. For conductors or superconductors the surface density of charge or current should be made explicit through appropriate mathematical transformations.

Some authors extend integrals to all space, in accordance to the concept that energy is contained in the fields. Although this seems to modify the value of F_{em} , it is not so because there are partial cancellations of the contributions of the fields inside the body with those outside it¹⁴¹. Special care should be taken with the calculation of the work necessary to polarize matter with fields assumed to be fixed, as sources provide or receive energy in order to keep fields constant when matter is introduced, even for very slow process¹⁴².

¹³⁸ Panofsky and Phillips, p. 90.

¹³⁹ Landau and Lifchitz, pp. 52-55, 129-131.

¹⁴⁰ Reitz, pp. 120 and 254.

¹⁴¹ Stratton, pp. 112-113, discusses this cancellation but gives no actual examples. See Problem 36.

¹⁴² See, for instance, V. Gilinsky and D. Holliday, *Interaction Energy of a Dielectric in an Electrostatic Field*, Am. J. Phys. vol. **34**, pp. 1134-1138 (1966).

F_{em} does not include the contribution from body's changes of shape, which is usually described using the thermodynamic potential called Gibbs free energy¹⁴³

$$d\Phi = U - T \cdot S + \mathbf{e} \cdot d\boldsymbol{\sigma}, \quad (5.10)$$

where work includes the variations of \mathbf{e} and stress takes the final value corresponding to the isothermal process. The variation of Gibbs free energy can be measured in actual experiments, but is more difficult to calculate than Helmholtz free energy.

For the formulas given in this chapter processes are assumed to be isothermal and Helmholtz free energy F is evaluated assuming a constant strain tensor \mathbf{e} . For instance, actual experiments should provide an efficient means of fast transference of any quantity of heat generated, so that the isothermal condition is fulfilled. These two conditions are seldom fulfilled in irreversible phenomena as hysteresis and in piezoelectric and piezomagnetic processes that modify significantly the dimensions of the crystal lattice, creating also additional fields. The use of the depolarization tensor method for these kind of phenomena—as well of others peculiar of single crystals—will probably require modifications or restrictions that have not been studied by the author¹⁴⁴.

Depending on the kind of anisotropy, it may be convenient to use coordinate axes different than the ellipsoid's semi-axes where \mathbf{N} is diagonal. The goal is to simplify the inverse matrices required for the case. This use of a coordinate system fixed to the body should take into account the case of rotating samples depicted in Figure 21, both for rotations and torques.

Solved problems related to the energy of polarizable ellipsoidal bodies immersed in applied fields are solved in Chapter 6. They are only selected examples of the way in which the depolarization tensor method may simplify the analysis of some experiments.

Anisotropy energy

A body under the action of an applied field experiences a torque if its energy depends on the orientation of the field respect to the body. A sphere of untextured polycrystalline iron has minimum free energy when \vec{M} and \vec{H}^0 are parallel. For any other orientation, the body's magnetic dipoles (that is, \vec{M}) experience a torque that tends to align them with the applied field. This does not generate a torque on the body when the material is isotropic, so that the magnetization may freely orientate with no expense of energy apart from dissipative effects. The sphere will remain in equilibrium at any orientation respect to the field because the magnetization will follow the field, not the body.

¹⁴³ Landau and Lifchitz, p. 62.

¹⁴⁴ Nye, chapter X pp. 170-191, makes a detailed discussion of the equilibrium thermodynamics of physical properties of single crystals. His figures 10.1a and 10.1b give a concise and quite illustrative summary of the problem.

The direction and magnitude of polarization —and therefore the free energy— may depend on the body’s orientation respect to the applied field. This anisotropy of energy has two different origins, shape and cristalline effects.

Shape anisotropy of energy¹⁴⁵ originates from the effect of the body’s shape on the internal field and polarization. For ellipsoidal bodies the effect is fully described by the internal depolarization tensor \mathbf{N} . For the family of ellipsoidal bodies, the only one without shape anisotropy is the sphere where the three eigenvalues are equal (see eq. 3.66). For the rest, the energy required to induce polarization along two different directions within the ellipsoid is in general different.

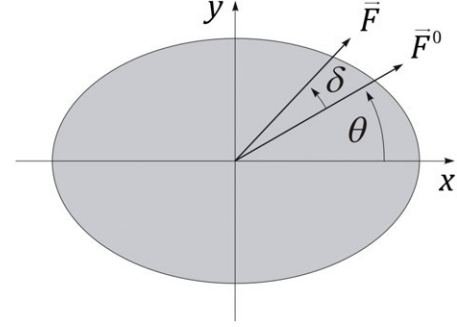


Figure 18. Deviation δ of the internal field respect to the applied one

The general equations that describe the phenomena of induced polarization are given in section 2.85 and reproduced below. The upper index that identifies the internal field has been suppressed because it will be the only field of interest in what follows.

$$\begin{aligned} \mathbf{F} &= \mathbf{F}^0 - \mathbf{N} \cdot \mathbf{Q}(\mathbf{F}), \quad \mathbf{Q}(\mathbf{F}) = \boldsymbol{\chi} \cdot \mathbf{F}, \\ \mathbf{F} &= (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi})^{-1} \cdot \mathbf{F}^0, \quad \mathbf{Q} = \boldsymbol{\chi} \cdot \mathbf{F}^{\text{int}} = \boldsymbol{\chi} \cdot (\mathbf{1} + \mathbf{N} \cdot \boldsymbol{\chi})^{-1} \cdot \mathbf{F}^0. \end{aligned} \quad (5.11)$$

Let’s analyze, then, the behaviour of an isotropic triaxial ellipsoidal body on the plane xy where the principal values of the internal depolarization tensor are N_a and N_b . The field and the polarization are exclusively those induced by the fix applied field \vec{F}^0 . In the body’s principal system of coordinates the components on the plane of interest are

$$F_y = v_y F_y^0, \quad F_z = v_z F_z^0, \quad v_\alpha = \frac{1}{1 + \chi N_\alpha}. \quad (5.12)$$

The deviation angle δ of the induced field respect to the applied one (see Figure 18) can be found from the scalar product

$$\begin{aligned} \vec{F} \cdot \vec{F}^0 &= \cos \delta \cdot F \cdot F^0 = F_y F_y^0 + F_z F_z^0 \\ &= v_y (F_y^0)^2 + v_z (F_z^0)^2 = (v_y \sin^2 \theta + v_z \cos^2 \theta) (F^0)^2. \end{aligned} \quad (5.13)$$

The modulus of the internal field differs from the applied one by a factor depending on the orientation of the latter respect to the body:

$$F = \sqrt{F_y^2 + F_z^2} = \sqrt{(v_y F_y^0)^2 + (v_z F_z^0)^2} = \sqrt{v_y^2 \sin^2 \theta + v_z^2 \cos^2 \theta} \cdot F^0. \quad (5.14)$$

¹⁴⁵ Brown (1963), p. 106.

The deviation angle δ is then given by

$$\cos \delta = \frac{\vec{F} \cdot \vec{F}^0}{F \cdot F^0} = \frac{v_y \sin^2 \theta + v_z \cos^2 \theta}{\sqrt{v_y^2 \sin^2 \theta + v_z^2 \cos^2 \theta}}. \quad (5.15)$$

\vec{F} and \vec{F}^0 are parallel only along the ellipsoid's principal axes, where its magnitude differs by the factor v_α . Due to shape anisotropy the minimum free energy is obtained when the body's largest or smaller semiaxis —depending on the kind of polarization, as discussed in next section— is oriented along the applied field. If this is not so, there is a torque that tends to orientate the body along that direction.

Crystalline anisotropy energy is originated by the spatial order of the material's atoms and molecules. A simple example that illustrates the phenomena is the two polarizable molecules analyzed in the section [Induced electric polarization of two interacting atoms](#). The only materials with crystalline anisotropy are single crystals or textured polycrystalline ones. In the latter the small crystals have some of its faces preferently oriented along certain direction, as happens in laminated metals. Powders, untextured polycrystalline materials and amorphous solids have no macroscopic order and, therefore, no crystalline anisotropy energy.

For the case of induced polarization —the only one discussed here—the symmetries that give origin to crystalline anisotropy are reflected in the non-scalar anisotropic susceptibility tensor. The main properties of this tensors are given in [Appendix 5](#). Its demonstration, not given there, can be made in similar way as for the case of the depolarization tensor (see section [Symmetries](#)) but requires a knowledge of crystalline structures that cannot be assumed for the readers of this book.

Origin of torques exerted on a body

In this section it is discussed the origin of the moment exerted on an ellipsoid outside the equilibrium orientation. The expression of the par resultante for each case is discussed in other sections.

At [Figure 19](#) an example is given of a common experiment where torque appears. A prolate spheroid, ellipsoidal shape commonly used in experiments, is suspended so that it can rotate around an axis normal to its symmetry axis x (corresponding to the largest semiaxis a). The rotation axis z is outgoing from the back of the figure. The minimum value of the isothermal and isobaric free energy F_e is assumed to be obtained when the direction of the largest semiaxis a coincides with that of the field \vec{F}_0 . If that semiaxis is initially an angle φ apart from the field, a couple is exerted on the body, such that tends to diminish it.

The couples's origin is a force distributed over the body's volume that, due to the constraint imposed by the rotation's axis, manifests as a moment of force along that axis. In the given configuration, where the plane xz normal to the page is a symmetry plane of the spheroid, the force over each differential element of volume has no component normal to plane yz . The appearance of this force is a common experimental fact for all potential energies as F_e . This is similar to what happens

with a mass, where the that tends to bring it to the position of minimum gravitational potential energy is the force distributed over all its volume that we call weight. It is discussed next the relationship between the force couple and the variation of the free energy.

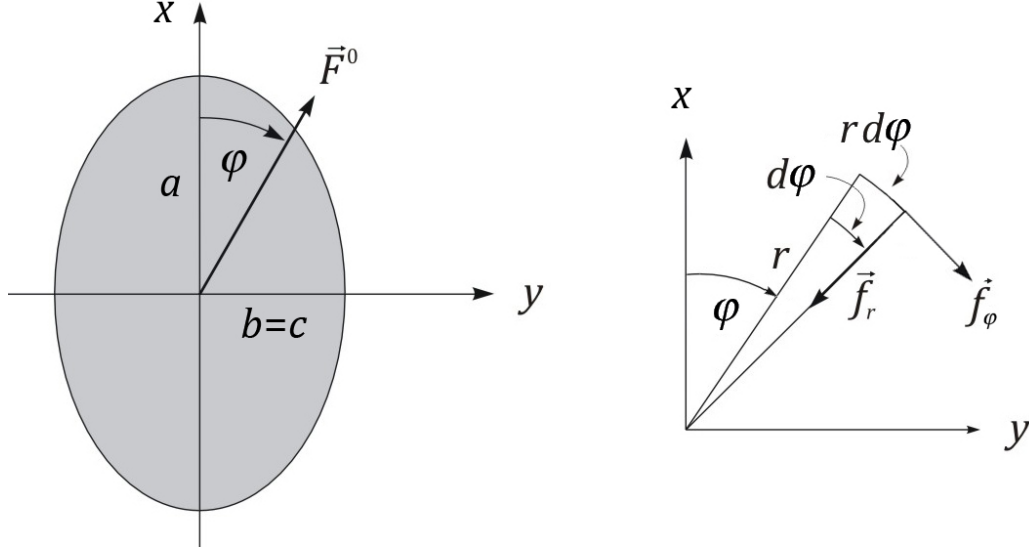


Figure 19. Torque exerted on ellipsoid in applied field.

At the right of the figure $\vec{f}(\vec{r})$ is the force over the element of volume dV at \vec{r} , where r is the distance to the rotation axis z . Angle φ is the usual one of spherical coordinates and the force has been decomposed in its tangential and radial components f_φ and f_r . The work dW done on the body when it rotates $d\varphi$ towards the direction of stable equilibrium is

$$dW = -d\varphi \iiint_V f_\varphi r d^3r = -\tau d\varphi = dF_e, \quad (5.16)$$

where τ is the torque exerted on the body along its suspension axis (see, for instance, eq. 5.23). Therefore

$$\tau = -\frac{dF_e}{d\varphi}. \quad (5.17)$$

Torque exerted on an ellipsoid and the state of equilibrium

In what follows a discussion is made of the problem of equilibrium and the force couples exerted on ellipsoidal bodies made with different kinds of materials and subject to static electric and magnetic fields.

Dielectrics

Permanent polarization

The analysis is made here of the equilibrium configuration of a permanent polarization in the absence of applied field. In such case the electrostatic contribution to the energy, eq. 5.9, becomes (see eq. 2.12)

$$F_e = \frac{V}{2\lambda} \vec{E}^P \cdot (\epsilon_0 \vec{E}^P + \lambda \vec{P}) = -\frac{\lambda V}{2\epsilon_0} \sum_{\alpha} N_{\alpha} (1 - N_{\alpha}) P_{\alpha}^2. \quad (5.18)$$

As an illustration, the minimization of energy will be analyzed for the case of spheroids, where

$$F_e = -\frac{\lambda V}{2\epsilon_0} (C_e P_e^2 + C_p P_p^2), \quad \text{where} \quad (5.19)$$

$$C_e = N_e (1 - N_e), N_p = 1 - 2N_e, C_p = 2N_e (1 - 2N_e),$$

where subindices p and e identify the polar and equatorial components or eigenvalues, respectively and the trace rule eqs. 3.15 was used. Figure 20 shows their values in a range that includes the right circular cylinder of infinite length (point A) to the sphere (point B). The sheet of constant thickness and infinite extension, not seen here, has the $N_p = 1$. The vertical line at $N_e = 1/3$ is the boundary of the regions of prolate and oblate spheroids.

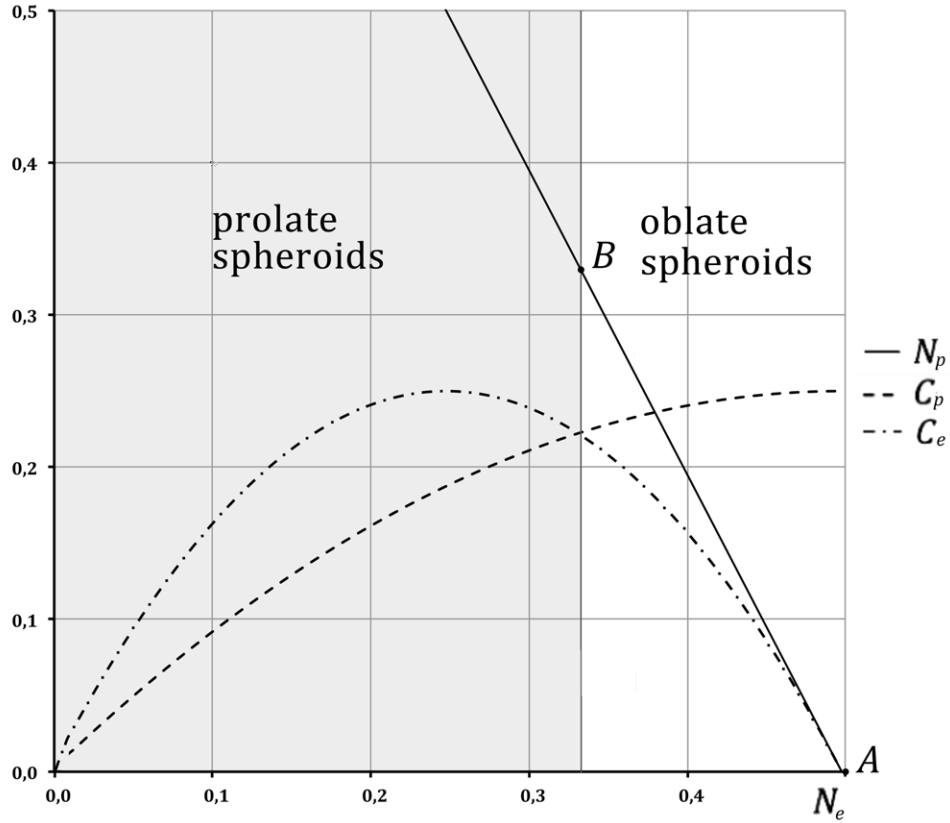


Figure 20. $N_{\parallel}, C_{\parallel}$ y C_{\perp} as functions of N_{\perp} .

For oblate spheroids the largest coefficient is C_p so that free energy reaches its maximum negative value when the polarization lies on the plane of largest semiaxes, the equatorial ones. For prolate spheroids the largest coefficient is C_e corresponding to the polar component (here the largest semiaxis), and the polarization tends to align along that direction. For the sphere the polarization has no preferred alignment because the coefficients are the same for all components of polarization.

The total free energy is not the expression customarily used for this sort of problem as it may be simplified eliminating the $\vec{E}^P \cdot \vec{E}^P$ (see, below, the analogous case of permanente magnetization), pero the obtained result is correct.

Induced polarization

According to Stratton¹⁴⁶, the torque $\vec{\tau}$ exerted in the linear case by the fixed field \vec{E}^0 on an isotropic dielectric body when its orientation is not the one of minimum free energy, is obtained from the following interaction energy,

$$F'_e = -\frac{1}{2} \iiint_V \vec{P} \cdot \vec{E}^0 d^3r = -\frac{V\epsilon_0\chi_e}{2\lambda} \sum_{\alpha} E_{\alpha}^{\text{int}} E_{\alpha}^0 = -\frac{V\epsilon_0\chi_e}{2\lambda} \sum_{\alpha} v_{\alpha} (E_{\alpha}^0)^2, \quad (5.20)$$

where $v_{\alpha} = \frac{1}{1 + \chi_e N_{\alpha}}$,

where eq. 2.33 was used. The moment of distributed forces is then¹⁴⁷ (see eq. 2.34):

$$\vec{\tau} = \vec{p} \times \vec{E}^0, \quad \vec{p} = \frac{\epsilon_0}{\lambda} \alpha_e \cdot \vec{E}^0, \quad \alpha_e = V \chi_e \cdot (1 + N \cdot \chi_e)^{-1}. \quad (5.21)$$

For an isotropic dielectric

$$\alpha_e = V \chi_e (1 + N \cdot \chi_e)^{-1} = V \chi_e \begin{pmatrix} \frac{1}{1 + \chi_e N_a} & 0 & 0 \\ 0 & \frac{1}{1 + \chi_e N_b} & 0 \\ 0 & 0 & \frac{1}{1 + \chi_e N_c} \end{pmatrix}. \quad (5.22)$$

When the applied field has no z component, as in Figure 19, the only torque component is

$$\tau_z = \frac{\epsilon_0}{\lambda} \chi_e^2 V \frac{N_c - N_b}{(1 + \chi_e N_b)(1 + \chi_e N_c)} \sin 2\varphi \vec{E}^0 \cdot \vec{E}^0. \quad (5.23)$$

The general case of spheroids is discussed at Problem 38. For triaxial ellipsoids with semiaxes order $a > b > c$, the order of the principal values is $N_c > N_b$ and the force couple has positive sign, corresponding to a rotation that tends to align the largest semiaxis with the field. This orientation is the one of minimum energy, as may be verified from the expression of F'_e initially written.

¹⁴⁶ Stratton, p. 113 eq. 51.

¹⁴⁷ Compare with Stratton, p. 216 eq. 53.

Anisotropic susceptibility

The electric susceptibility of uniaxial crystals has—in its principal system of coordinates, that in general does not coincide with that of the body—the following expression (see [Appendix 5](#)):

$$\chi_e = \begin{pmatrix} \chi_{\perp} & 0 & 0 \\ 0 & \chi_{\perp} & 0 \\ 0 & 0 & \chi_{\parallel} \end{pmatrix}. \quad (5.24)$$

The microscopic origin of this anisotropy is not discussed here, but has a similar origin as that of the two polarizable atoms analyzed at page [21](#), with the additional complexity introduced by the regular order of a crystal lattice.

The force couple exerted on a spherical body of that material—that is, without shape anisotropy—does not determine the two components but its difference, as discussed next

From eqs. [2.33](#) and [3.66](#), the components of the internal field and the polarization in the principal system of coordinates of the susceptibility tensor are:

$$E_{\alpha}^{\text{int}} = \frac{E_{\alpha}^0}{1 + \chi_{\alpha}/3}, \quad P_{\alpha} = \chi_{\alpha} E_{\alpha}^{\text{int}} = \frac{\chi_{\alpha} E_{\alpha}^0}{1 + \chi_{\alpha}/3}. \quad (5.25)$$

Because of the rotational symmetry of the susceptibility, the experiment must be done on the plane that contains the symmetry axis and the applied field. When descomposing all vectores in their components parallel and perpendicular to that axis, the internal field is give by

$$E_{\perp}^{\text{int}} = \frac{E_{\perp}^0}{1 + \chi_{\perp}/3}, \quad E_{\parallel}^{\text{int}} = \frac{E_{\parallel}^0}{1 + \chi_{\parallel}/3}. \quad (5.26)$$

Upon replacement in the expression for the free energy eq. [5.20](#) it is obtained

$$\begin{aligned} F'_e &= -\frac{1}{2} \iiint_V \vec{P} \cdot \vec{E}^0 d^3r = -\frac{\epsilon_0}{2\lambda} V \sum_{\alpha} \chi_{\alpha} E_{\alpha}^{\text{int}} E_{\alpha}^0 \\ &= -\frac{\epsilon_0}{2\lambda} V \left[\chi_{\perp} \frac{(E_{\perp}^0)^2}{1 + \chi_{\perp}/3} + \chi_{\parallel} \frac{(E_{\parallel}^0)^2}{1 + \chi_{\parallel}/3} \right] = -\frac{\epsilon_0}{2\lambda} V \left[\frac{(E_{\perp}^0)^2}{1/\chi_{\perp} + 1/3} + \frac{(E_{\parallel}^0)^2}{1/\chi_{\parallel} + 1/3} \right]. \end{aligned} \quad (5.27)$$

F'_e is minimum along the directions with maximum susceptibility. For $\chi_{\parallel} > \chi_{\perp}$ the minimum is obtained when the symmetry axis is parallel to \vec{E}^0 . For $\chi_{\parallel} < \chi_{\perp}$, the minimum is obtained when the symmetry axis is normal to the field. Although at first sight this appears indistinguishable from a shape anisotropy, it is not so (see [Problem 02](#)). The calculation of the torque exerted by the applied field on the body is done at [Problem 39](#).

Magnetic materials

Permanent magnetization

The magnitude of the spontaneous magnetization is usually constant at constant temperature, independently of its orientation respect to the crystal axes^{148,149}. One may then compute the energy of spontaneously magnetized matter assuming invariable the magnitude of the microscopic magnetic moments. This calculation—which does not include the energy necessary for building up the magnetic atomic and molecular moments— gives^{150,151}

$$F_m = -\frac{\mu_0}{2} \iiint_V \vec{H}^{\text{int}}(\vec{r}) \cdot \vec{M}(\vec{r}) d^3r. \quad (5.28)$$

For a uniformly magnetized ellipsoidal body eq. 2.44 gives

$$\vec{H}^{\text{int}} = -\lambda' \vec{N} \cdot \vec{M}. \quad (5.29)$$

Therefore

$$F_m = -\frac{\mu_0}{2} V \vec{H}^{\text{int}} \cdot \vec{M} = \frac{\mu_0 \lambda'}{2} V \vec{M} \cdot \vec{N} \cdot \vec{M}, \quad (5.30)$$

expression that should be minimized with the condition of constant M . In the principal system of coordinates of \vec{N} the spherical components of \vec{M} are

$$M_x = M \sin \theta \cos \varphi, \quad M_y = M \sin \theta \sin \varphi, \quad M_z = M \cos \theta. \quad (5.31)$$

Therefore

$$\begin{aligned} F_m &= \frac{\mu_0 \lambda'}{2} V \sum_{\alpha} N_{\alpha} M_{\alpha}^2 \\ &= \frac{\mu_0 \lambda'}{2} V \left(N_x \sin^2 \theta \cos^2 \varphi + N_y \sin^2 \theta \sin^2 \varphi + N_z \cos^2 \theta \right) M^2, \end{aligned} \quad (5.32)$$

should be minimized.

The minimum is easily found realizing that the parenthesis is the parametric equation

$$X = \sqrt{N_x} \sin \theta \cos \varphi, \quad Y = \sqrt{N_y} \sin \theta \sin \varphi, \quad Z = \sqrt{N_z} \cos \theta, \quad (5.33)$$

of the ellipsoid

$$\frac{X^2}{(\sqrt{N_x})^2} + \frac{Y^2}{(\sqrt{N_y})^2} + \frac{Z^2}{(\sqrt{N_z})^2} = 1, \quad (5.34)$$

¹⁴⁸ Kittel, p. 533.

¹⁴⁹ Landau and Lifchitz, p. 146.

¹⁵⁰ Chikazumi, p. 24 eq. 1.95.

¹⁵¹ Stratton, p. 130 eq. 63.

whose square radius vector R^{152} has to be maximized. The minimum value of R^2 is obtained along the axis corresponding to the smallest denominator, that is, the direction of the ellipsoid's largest semiaxis (see eq. 4.6).

This property of permanently magnetized prolate spheroids explains the stability of the magnetization of a compass's needle. In this case there is no torque applied on the body, but an irreversible rotation of magnetization (hysteresis) that aligns it in the direction of minimum isothermal and isobaric free energy. Such a body has a permanent magnetic dipole moment given among eqs. 2.49 and its direction is that of the polar semiaxis. At constant temperature the magnetization of the spheroid may be considered to be fixed because in the presence of low magnetic fields like the terrestrial one, the induced magnetization is much lower than the spontaneous or permanent one¹⁵³. For such a case¹⁵⁴ the free energy of interaction with an applied field and the force couple are

$$F_m = -\vec{m} \cdot \vec{B}^0, \quad \vec{\tau} = \vec{m} \times \vec{B}^0. \quad (5.35)$$

These expressions do not required further análisis, as F_m is a minimum when \vec{m} has the same direction and sense than \vec{B}^0 , orientation generated by $\vec{\tau}$ that vanishes when both vectors are parallel.

Induced magnetization

For the calculation of the interaction energy of spontaneous electric or magnetic polarizations it is not necessary to include the energy of its creation. It is not so for induced polarizations, where the required expression of the free energy is¹⁵⁵

$$F_m = -\frac{\mu_0}{2} \iiint_V \vec{H}^0(\vec{r}) \cdot \vec{M}(\vec{r}) d^3r = -\frac{\mu_0}{2} V \vec{M} \cdot \vec{H}^0 = -\frac{1}{2} \vec{m} \cdot \vec{B}^0, \quad (5.36)$$

because both vectors are uniform (see page 6).

The torque exerted by the magnetic induction field over the ellipsoidal body is then given by (see eq. 2.49)

$$\vec{\tau} = \vec{m} \times \vec{B}^0, \quad \vec{m} = \frac{1}{\lambda'} \alpha_m \cdot \vec{H}^0, \quad \alpha_m = V \chi_m \cdot (1 + N \cdot \chi_m)^{-1}. \quad (5.37)$$

¹⁵² Radius vector is the distance of a surface's point to its center.

¹⁵³ Chikazumi, p. 439.

¹⁵⁴ Jackson, p. 150. The factor 1/2 is not present when the moment and the field are "rigid" or fixed.

¹⁵⁵ Stratton pp. 128 eq. 51 and 154 eq. 7. An illuminating discussion of the subject is made by M. S. Plesset and G. Venezian (Am. J. Phys. vol. **32**, pp. 860-864, 1964), where the calculation is made of the torque applied on an ellipsoidal body with induced magnetization. See also S. P. Puri, Am. J. Phys. vol. **33**, p. 472, 1965.

Magnetic torque experiments

Figure 21 shows the main parts of a goniometer, device that can measure with good precision the force couples exerted by an applied magnetic field over a magnetized body. The instrument is based on a calibrated torsion wire that can be rotated in order to return a reflected light beam to its zero (position in the absence of applied field). The glass tube blocks air currents that may displace the body. The paddles immersed in oil are adjusted in size and shape in order to provide critical damping for a fast return to zero.

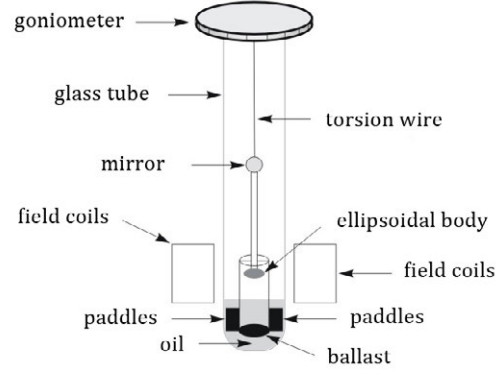


Figure 21. Elementary measurement of magnetic force couples.

For the isotropic case the force couple reduces to

$$\vec{\tau} = \vec{m} \times \vec{B}^0 = \frac{\mu_0}{\lambda'} V \chi_m \vec{H}^0 \cdot (\mathbf{1} + \chi_m \mathbf{N})^{-1} \cdot \vec{H}^0. \quad (5.38)$$

In the ellipsoid's principal system of coordinates the components are given by

$$\tau_x = \frac{\mu_0 \chi_m^2 V}{\lambda'} \left(\frac{N_c - N_b}{(1 + \chi_m N_b)(1 + \chi_m N_c)} \right) H_y^0 H_z^0, \quad (5.39)$$

$$\tau_y = \frac{\mu_0 \chi_m^2 V}{\lambda'} \left(\frac{N_a - N_c}{(1 + \chi_m N_a)(1 + \chi_m N_c)} \right) H_x^0 H_z^0, \quad (5.40)$$

$$\tau_z = \frac{\mu_0 \chi_m V}{\lambda'} \left(\frac{N_b - N_a}{(1 + \chi_m N_a)(1 + \chi_m N_b)} \right) H_x^0 H_y^0. \quad (5.41)$$

Both for the paramagnetic ($\chi_m > 0$) and the diamagnetic ($\chi_m < 0$, $|\chi_m| \ll 1$) case denominators are positive numbers and the sign is determined by the difference of principal values. When the applied field has only y, z components, the treatment is similar to that given for the electric one, eq. 5.23. The torque tends then to align the largest semiaxes (a in our convention) with the applied field, both in the paramagnetic and diamagnetic case.

Conductors

The energy of a surface charge distribution on a conductor with net charge Q is given by the next equation¹⁵⁶.

¹⁵⁶ Stratton, p. 107 eq. 16.

$$F_e = \frac{1}{2\lambda} \oint\oint_S \phi \sigma(\vec{r}) d^2r = \frac{1}{2\lambda} \phi_s \oint\oint_S \sigma(\vec{r}) d^2r = \frac{1}{2\lambda} \phi_s Q = 0. \quad (5.42)$$

The result is obtained by computing the work done by bringing the charges from infinite distances to a conductor at potential ϕ_s until the total charge Q is build up. The formula is not valid for the separation of charges over the surface of an ellipsoid with zero net charge. Classical texts on electromagnetism, like those of Stratton and Jackson, restrict their analysis to conducting bodies with non vanishing net charge where the previously given shape-independent formula is valid. The exception is that of Landau and Lifshitz, who compute the following value for the interaction energy of a zero net charge conductor with an applied electrostatic field:

$$F_e = -\frac{1}{2} \vec{p} \cdot \vec{E}^0, \quad (5.43)$$

where \vec{p} is the electric dipole moment of the body. For ellipsoidal conductors this moment is given by eq. 2.60, so that in the principal system of coordinates

$$F_e = -\frac{1}{2} \vec{E}^0 \cdot \vec{p} = -\frac{\epsilon_0 V}{2\lambda} \vec{E}^0 \cdot \mathbf{N}^{-1} \cdot \vec{E}^0 = -\frac{\epsilon_0 V}{2\lambda} \sum_{\alpha} \frac{1}{N_{\alpha}} (E_{\alpha}^0)^2, \quad (5.44)$$

coinciding with eq. 2.14 of Landau & Lifshitz.

The expression for the force couple exerted over a conducting ellipsoid is the one for a dielectric, eq. 5.21, in the limit of infinite susceptibility (see Problem 08). It is thus obtained

$$\tau_x = \frac{\epsilon_0 V}{\lambda} \frac{N_c - N_b}{N_b N_c} E_y^0 E_z^0, \quad \tau_y = \frac{\epsilon_0 V}{\lambda} \frac{N_a - N_c}{N_a N_c} E_x^0 E_z^0, \quad \tau_z = \frac{\epsilon_0 V}{\lambda} \frac{N_b - N_a}{N_a N_b} E_x^0 E_y^0. \quad (5.45)$$

When the three depolarization factors are equal—that is, for spheres—the force couple vanishes, as corresponds to its shape isotropy. When all three are different, as in triaxial ellipsoids, the force couple tends to align the largest semiaxis with the field, as happens in the dielectric and magnetic case.

Superconductors

As in the case of conductors, the force couple exerted by the applied field on a superconducting body may be derived from the case of induced magnetization, eq. 5.37, using the condition of perfect diamagnetism $\chi_m = -1$ (see eq. 2.73).

$$\vec{\tau} = \vec{m} \times \vec{B}^0, \quad \vec{m} = \frac{1}{\mu_0 \lambda'} \alpha_s \cdot \vec{B}^0, \quad \alpha_s = -V(\mathbf{1} - \mathbf{N})^{-1}. \quad (5.46)$$

A typical component is

$$\tau_x = \frac{V}{\mu_0 \lambda'} \left(\frac{N_z - N_y}{(1 - N_y)(1 - N_z)} \right) B_y^0 B_z^0, \quad (5.47)$$

where the other components are obtained from a cyclic permutation of indices.

For finite volume ellipsoids the denominator is always positive and the behaviour of a superconductor body is the same as that of a magnetized one, with the minimum energy obtained when the largest semiaxis is aligned with the applied field. Degenerate ellipsoids —as the sheet of infinite extension and the elliptic cylinders— are not valid cases for the analysis of torques that there diverge with volume.

Force on an ellipsoid in a non-uniform field

The depolarization tensor method has few advantages for the treatment of the forces exerted by non-uniform fields on ellipsoids. Nevertheless, due to its experimental relevance, the expressions are given of the forces exerted on polarized dielectric and magnetic ellipsoids in quasi-uniform fields^{157,158}:

$$\begin{aligned}\vec{f}_e &= V(\vec{P} \cdot \nabla) \vec{E}^0(\vec{r}) = (\vec{p} \cdot \nabla) \vec{E}^0(\vec{r}) = \frac{\epsilon_0}{\lambda} (\vec{E}^0(\vec{r}) \cdot \alpha_e \cdot \nabla) \vec{E}^0(\vec{r}), \\ \vec{f}_m &= V(\vec{M} \cdot \nabla) \vec{B}^0(\vec{r}) = (\vec{m} \cdot \nabla) \vec{B}^0(\vec{r}) = \frac{1}{\mu_0 \lambda'} (\vec{B}^0(\vec{r}) \cdot \alpha_m \cdot \nabla) \vec{B}^0(\vec{r}).\end{aligned}\tag{5.48}$$

where the body's polarizability tensors, eqs. 2.33 and 2.49, have been used.

The variation of the field inside the body should be small enough so that the depolarization tensor method is valid within the acceptable range of error. At the same time, the variation should be large enough so that the total force is measurable. Experiments of this type¹⁵⁹ make possible measurements of diamagnetic susceptibility that are impossible with the torsion method (see *Magnetic torque experiments* at page 112). This is so because of the small value of this kind of susceptibility and the fact that torques are proportional to its square; force, on the other hand, is directly proportional to the susceptibility.

Infinite and infinitesimal bodies

Equation 3.80 shows that similar ellipsoids have equal internal depolarization tensors \mathbf{N} . Therefore, if one makes all semiaxes grow without limit preserving their relative values, \mathbf{N} does not change. One must then be careful with the use of bodies of infinite or indefinite extension, subterfuge often used for the introduction of dielectric or magnetic material surrounding finite bodies, unless one specifies their shape. These idealizations are justified only if they provide a good approximation to a real situation, that usually involves finite bodies. The sphere of infinite diameter can be used to approximate an isotropic dielectric environment, but one has still to discuss the effect of the cavity that contains the finite body of interest. The elliptic cylinder of infinite length provides a good approximation to the more

¹⁵⁷ Landau and Lifchitz, p. 72 eq. 16.12.

¹⁵⁸ Landau and Lifchitz, p. 144 eq. 34.8.

¹⁵⁹ Faraday's method, see H. Zijlstra, *Experimental Methods in magnetism*, vol. 2, North Holland Publishing, 1967, p. 94.

complex calculation of the field near the equator of a triaxial ellipsoid, but there are better approximations than the sheet of infinite extension to the field near the poles of a very flat oblate spheroid (see eq. 4.29).

Polarized matter creates fields in two different ways: by the space variation of polarization inside the body's volume and by the step discontinuity through the body's surface¹⁶⁰. For uniform polarizations as the ones studied here, only the second effect appears, as illustrated for the electric case by eq. 2.51,

$$\vec{E}(\vec{r}) = \vec{E}^0 - k_1 \nabla \oint_S \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2 r'.$$

The contribution to the field of surface element dS is $\sigma dS/r^2 = \sigma d\Omega$ (notice that the gradient increases in 1 the exponent of the denominator), where $d\Omega$ is the subtended element of solid angle. Therefore, the contribution of a surface sector to the field at an internal point is proportional to the subtended solid angle. This is illustrated by the infinite sheet and elliptic cylinder, where the depolarization factors (principal values) vanish for the directions where the solid angle tends to 0 (the semiaxis tends to ∞). That is, very large distances do not ensure very small contributions from the surface sector to the field, that depends on the solid angle subtended by that sector. Although at large distances r the field of a finite polarized body, whatever its shape, has a distance dependence of r^{-3} (see section [Near and far away point approximations to next](#)), far away surface charges do not have this behaviour.

From the mathematical point of view the problems with bodies of infinite extension originate when the value of the integral

$$\iiint_V \frac{d^3 r'}{|\vec{r} - \vec{r}'|} \quad (5.49)$$

is computed by extending V over all space. A careful analysis shows that the integral is semi-convergent, that its value when volume V grows depends on the peculiar way the infinite limit is taken¹⁶¹. An infinite number of different limits may exist depending on the shape of V , as illustrated by the property eq. 3.80 of the depolarization tensor.

The same error of assuming all infinite bodies to be equal has been made with infinitesimal ones. The Fermi contact term is a little known contribution to the energy of atomic electrons originated in its interaction with the nucleus. Its calculation was made considering the nucleus to be a point charge, implicitly spherical. The term contains an integral that this author proved to be of the internal depolarization tensor type¹⁶², thus requiring a better specification of the nuclear charge distribution. Similar errors are repeatedly made in the calculations

¹⁶⁰ Reitz, p. 79 eqs. 4.13 and 4.14.

¹⁶¹ MacMillan, pp. 163-165.

¹⁶² C. E. Soliv  rez; [The contact hyperfine interaction: an ill defined problem](#); J. Phys. C: Solid St. Phys. vol. 13; pp. L1017- L1019; 1980.

of microscopic fields in crystals when series of dipolars are numerically summed without analyzing its kind of convergence.

Cavities

Standard treatment

Cavities have had a distinguished role in the theory of electricity since Lord Kelvin used them to define \vec{E} and \vec{D} ¹⁶³. The problem is that they are treated as ellipsoidal bodies with their same properties¹⁶⁴, or as if the extraction of an ellipsoidal piece of a uniformly polarized body does not modify the uniform internal fields. Stratton, without previous discussion, treats cavities as if they were bodies of the same shape with the dielectric permittivity of vacuum¹⁶⁵. Another reputed author is more explicit when he states¹⁶⁶:

*Finally we consider the field inside a cavity in a material magnetized to an intensity I (Fig. 1.16). The free pole distribution on the surface of the cavity is the same as that on the surface of a solid body with the same shape as the cavity, and with the same magnetization as the material surrounding the cavity, except that the poles are of opposite sign. **This must be true, because if we superpose the body and the hole, we have a uniformly magnetized solid without free poles.***

As previously stated the argument in bold type is true for uniform polarization, where the only contribution to the field comes from surface polarization charges¹⁶⁷. But this can only be equal to the polarization on the cavity's walls when the polarization is fixed (*rigid*, as some authors call it), which is false.

Jackson explicitly solves the spherical cavity¹⁶⁸ but he replaces in the solution for the sphere the electric permittivity by its inverse with the pseudo-argument:

In fact, inspection of boundary conditions (4.56) shows that the results from the cavity can be obtained from those for the sphere by the replacement $\epsilon \rightarrow 1/\epsilon$.

The alluded boundary conditions are the continuity of the tangential component of the electric field through the cavity's surface and the step discontinuity of its normal component due to the polarization charges eq. 3.35. In addition, he does not specify the shape of the dielectric body where the cavity is made.

Van Vleck¹⁶⁹, a reputed specialist on the properties of polarized matter, makes use of cavities for the calculation of local fields, but his cavities are virtual, not real, a mere artifact for promoting fruitful discussions. This is the case of the so called Lorentz sphere, "cavity" used to estimate the internal or local electric field at the

¹⁶³ See, for instance, Stratton, p. 214; Reitz, p. 82.

¹⁶⁴ Stratton, pp. 206 and 213.

¹⁶⁵ Stratton, p. 206 eq. 31.

¹⁶⁶ Chikazumi, p. 16.

¹⁶⁷ Reitz, p. 79 eqs. 4.13 and 4.14.

¹⁶⁸ Jackson, pp. 114-115.

¹⁶⁹ Van Vleck, chapter IV, analyzes classical calculations of fields inside cavities.

molecules of a dielectric¹⁷⁰, magnitude different from the macroscopic field in Maxwell's equations. The "cavity" is not empty but filled with molecules whose field and polarization are to be computed individually according to their known crystalline positions.

The analysis made at section *Induced electric polarization of two interacting atoms* illustrates the mutual influence of different portions of matter on the final polarization state. Polarization is not automatically uniform, nor it remains forever fixed when this state is achieved. Shape effects are crucial and only closed single surfaces of the second degree (ellipsoids) have been proved to be amenable of uniform polarization. It is therefore clear that the following two aspects have to be carefully analyzed:

1. The effect of the removal of material inside the ellipsoidal body on the polarization state of the rest of the body.
2. The effect of the polarization charges or currents on the cavity's walls over the fields inside the cavity and the remaining polarized matter.

It should also be stressed that a uniform applied and total field inside the body is required, but this is a necessary condition and not a sufficient one.

Some specific cavities: or homoeoids

A cavity is the result of the remotion of a volume V_0 of polarized matter from a larger one V . The simplest assumption is that both pieces have ellipsoidal shape, so that they can, in principle, be uniformly polarized. The integral from which the depolarization tensor is to be calculated (see the definition eq. 3.4) is

$$I'(\vec{r}) = \iiint_{V-V_0} \frac{d^3r'}{|\vec{r}-\vec{r}'|} = \iiint_V \frac{d^3r'}{|\vec{r}-\vec{r}'|} - \iiint_{V_0} \frac{d^3r'}{|\vec{r}-\vec{r}'|} = I(\vec{r}) - I_0(\vec{r}), \quad (4.50)$$

where a well known mathematical property of integrals was used. Using the definition eq. 3.4 one may derive from it the following mathematical relationship

$$\mathbf{n}'(\vec{r}) = \mathbf{n}(\vec{r}) - \mathbf{n}_0(\vec{r}), \quad (5.51)$$

where \mathbf{n} is the depolarization tensor of the ellipsoidal body of volume V and \mathbf{n}_0 the one of V_0 . This seems to reduce the problem of fields to a an algebraic combination of known cases, but it is not so. The superposition principle applies to rigid distributions of charges, dipoles or currents, but not to induced polarizations. Although this should be clear enough, it is best to solve some specific case to illustrate the argument, what is done next.

Shells or homoeoids are one of the most symmetric case of cavities, when the removed ellipsoidal region V_0 is concentric to a similar ellipsoid with proportional corresponding semiaxes. If the first of the equations below describes the body's surface and the second one the surface of the cavity, they determine an homoeoid or shell:

¹⁷⁰ Dekker, pp. 141-144.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k^2, \quad \text{where } 0 < k < 1. \quad (5.52)$$

Homoeoids have been intensively studied in the theory of gravitational¹⁷¹ and electromagnetic potentials because they have the interesting property that the potential is constant inside the shell, that is, internal fields vanish as inside conductors but this may not apply to polarization fields.

Problem 28 solves the simple case of an spherical shell using the electrostatic Gauss's Law. The values of the auxiliary function $f(\vec{r})$ given by eq. 6.72 show that in none of the three regions depicted by Figure 24 it is a quadratic function of coordinates. Therefore, nowhere may a depolarization tensor be defined and, presumably, no uniform internal field and polarization may be found. This does not rule out the possibility that the field inside the cavity is uniform, it only shows that the depolarization tensor method is not the appropriate one to make the calculation.

If its cavities are small enough, the polarization of an allipsoidal body will have small departures from that calculated from its depolarization tensor. The reason is that the change in the field external to the cavity is proportional to its dipole moment, that is to the cavity's volume. The field around the cavity, on the other hand, would be very different from the one in its absence. The most common error made with the treatment of cavities is the assumption that the polarization of the surrounding matter remains inalterable, fixed, which is false without doubt.

Thin shells and the dipole layer

The lack of uniformity in the body's interior does not appear in an infinitesimally thin shell, where the body is reduced to a double sided surface, case that is now analyzed. The case of the spherical homoeoid is analyzed here (see problem Problem 28), where for the three different regions it is obtained

$$\frac{\partial^2 f_1}{\partial x_\alpha \partial x_\beta} = 0 \quad \text{for } 0 \leq r \leq R_1, \quad (5.53)$$

$$\left(\frac{\partial^2 f_2}{\partial x_\alpha \partial x_\beta} \right) = \begin{pmatrix} \frac{1}{3} + \frac{R_1^3}{3} \frac{3x^2 - r^2}{r^5} & R_1^3 \frac{x \cdot y}{r^5} & R_1^3 \frac{x \cdot z}{r^5} \\ R_1^3 \frac{y \cdot x}{r^5} & \frac{1}{3} + \frac{R_1^3}{3} \frac{3y^2 - r^2}{r^5} & R_1^3 \frac{y \cdot z}{r^5} \\ R_1^3 \frac{z \cdot x}{r^5} & R_1^3 \frac{z \cdot y}{r^5} & \frac{1}{3} + \frac{R_1^3}{3} \frac{3z^2 - r^2}{r^5} \end{pmatrix} \quad \text{for } R_1 \leq r \leq R_2, \quad (5.54)$$

¹⁷¹ A. Gray, *On the Attractions of Spherical and Ellipsoidal Shells*, Proc. Edinburgh Math. Soc. vol. 23, pp. 91-100 (1913-1914).

$$\begin{aligned}
& \left(\frac{\partial^2 f_3}{\partial x_\alpha \partial x_\beta} \right) \\
& = -\frac{R_2^3 - R_1^3}{3} \begin{pmatrix} \frac{3x^2 - r^2}{r^5} & \frac{3x \cdot y}{r^5} & \frac{3x \cdot z}{r^5} \\ \frac{3y \cdot x}{r^5} & \frac{3y^2 - r^2}{r^5} & \frac{3y \cdot z}{r^5} \\ \frac{3z \cdot x}{r^5} & \frac{3z \cdot y}{r^5} & \frac{3z^2 - r^2}{r^5} \end{pmatrix} \quad \text{for } r \geq R_2.
\end{aligned} \tag{5.55}$$

Taking the limit $R_2 \rightarrow R_1$ in eqs. 5.54 and 5.55 it is obtained

$$\mathbf{n}^{\text{cav}} = 0, \quad \mathbf{N} = \begin{pmatrix} \frac{x^2}{R_1^2} & \frac{xy}{R_1^2} & \frac{xz}{R_1^2} \\ \frac{yx}{R_1^2} & \frac{y^2}{R_1^2} & \frac{yz}{R_1^2} \\ \frac{zx}{R_1^2} & \frac{zy}{R_1^2} & \frac{z^2}{R_1^2} \end{pmatrix} = \hat{\mathbf{s}}(\vec{r}^{S_1}) \hat{\mathbf{s}}(\vec{r}^{S_1}), \quad \mathbf{n}^{\text{ext}} = 0. \tag{5.56}$$

The expression, valid for all thin ellipsoidal homoeoids (see [Problem 42](#)), shows that the depolarization tensor of a thin shell is reduced to the discontinuity through the ellipsoid's surface eq. 3.38. Its utility has yet to be assessed.

Chapter 6: Selected problems

Electric polarization

Problem 01: Autoconsistent electric polarization of two atoms

Two identical and isotropic atoms, with no permanent electric dipole moment, are a distance d apart and immersed in a uniform electric applied field. Assuming that the induced dipole moments are proportional to the total electric field at each atom, calculate their values using the point dipole model.

Solution

See section *Induced electric polarization of two interacting atoms*.

Problem 02: Shape or crystalline anisotropy?

In a laboratory, a student made an experimental determination of the two coefficients $c_{\perp} \neq c_{\parallel}$ that relate the electric polarization with the applied field for an ellipsoid:

$$\frac{\lambda}{\epsilon_0} P_{\perp} = c_{\perp} E_{\perp}^0, \quad \frac{\lambda}{\epsilon_0} P_{\parallel} = c_{\parallel} E_{\parallel}^0. \quad (6.1)$$

The absent minded student that made the measurements forgot to register to which of the two following bodies they correspond: 1) an isotropic dielectric spheroid; 2) a uniaxial dielectric sphere (see [Appendix 5](#) at page 165). A large amount of time, work and resources were used, so it is highly desirable to find a way to identify the right body. Is it possible to differentiate between the two cases without repeating the measurements?

Solution

In the principal system of coordinates of a spheroid, the electric polarization of an isotropic material with susceptibility χ placed in a uniform field has the following components (see eq. 2.33)

$$\frac{\lambda}{\epsilon_0} P_{\perp} = \frac{\chi}{1 + N_{\perp} \chi} E_{\perp}^0 = c_{\perp} E_{\perp}^0, \quad \frac{\lambda}{\epsilon_0} P_{\parallel} = \frac{\chi}{1 + N_{\parallel} \chi} E_{\parallel}^0 = c_{\parallel} E_{\parallel}^0. \quad (6.2)$$

As the values N_{\perp}, N_{\parallel} of the spheroid may be easily computed, each equation provide a value of χ ,

$$\chi_{\perp} = \frac{c_{\perp}}{1 - N_{\perp} c_{\perp}}, \quad \chi_{\parallel} = \frac{c_{\parallel}}{1 - N_{\parallel} c_{\parallel}}. \quad (6.3)$$

If $\chi_{\perp} = \chi_{\parallel} = \chi$ the problem is solved.

Otherwise the expressions for a uniaxial dielectric sphere have to be solved. In the susceptibility tensor's principal system the equations are

$$\frac{\lambda}{\epsilon_0} P_{\perp} = \frac{\chi_{\perp}}{1 + \chi_{\perp}/3} E_{\perp}^0 = c_{\perp} E_{\perp}^0, \quad \frac{\lambda}{\epsilon_0} P_{\parallel} = \frac{\chi_{\parallel}}{1 + \chi_{\parallel}/3} E_{\parallel}^0 = c_{\parallel} E_{\parallel}^0. \quad (6.4)$$

The eigenvalues are

$$\chi_{\perp} = \frac{c_{\perp}}{1 - c_{\perp}/3}, \quad \chi_{\parallel} = \frac{c_{\parallel}}{1 - c_{\parallel}/3}, \quad (6.5)$$

where the coefficients c cannot be equal. The susceptibility's eigenvalues have the same order as the coefficients.

Therefore, the shape anisotropy of a spheroid has some characteristics that resemble the dielectric anisotropy of a sphere, but others are different.

Problem 03: Dielectric sphere

Give, in SI units, the expressions of the internal and external electric field of a dielectric sphere under an applied field. Compute the body's electric dipole moment and compare the results with those obtained by Panofsky and Phillips (PP) by solving Laplace's differential equation¹⁷². Identify the *depolarization factor* there given and verify if it has the right value.

Solution

For a sphere of radius a , in SI units, eqs. 2.33 and A1.14 give

$$E_z^{\text{int}} = \frac{3}{(3\epsilon_0 + \epsilon_0\chi)/\epsilon_0} E^0 = \frac{3}{\kappa + 2} E^0, \quad (6.6)$$

where $\kappa = 1 + \chi$ is the dielectric constant.

From the potential PP, eq. 5-18, it is obtained

$$\phi(\vec{r}) = -\frac{3E^0}{\kappa + 2} z, \quad E_z = -\frac{\partial\phi(\vec{r})}{\partial z} = \frac{3}{\kappa + 2} E^0 \quad \text{for } r < a, \quad (6.7)$$

coincident con the previous equation.

For the depolarization tensor method the electric dipole moment is related to the polarization by eqs. 2.33:

$$\vec{P} = \epsilon_0 \chi \vec{E}_z^{\text{int}} = \frac{3\epsilon_0\chi}{\kappa + 2} \vec{E}^0 = \frac{3\epsilon_0(\kappa - 1)}{\kappa + 2} \vec{E}^0, \quad \vec{p} = V\vec{P} = 4\pi\epsilon_0 a^3 \frac{\kappa - 1}{\kappa + 2} \vec{E}^0, \quad (6.8)$$

¹⁷² Panofsky and Phillips, pp. 76-77.

the same value given by PP's eq. 5-19.

The expression for the external field in the depolarization tensor method is given by eqs. 2.33:

$$\vec{E}^{\text{ext}}(\vec{r}) = \vec{E}^0 - \frac{1}{\varepsilon_0} \vec{n}^{\text{ext}}(\vec{r}) \cdot \vec{P} = \vec{E}^0 - 3 \frac{\kappa-1}{\kappa+2} \vec{n}_{\text{sphere}}^{\text{ext}}(\vec{r}) \cdot \vec{E}^0, \quad (6.9)$$

where $\vec{n}_{\text{sphere}}^{\text{ext}}$ is given by eq. 3.66. Thus

$$\begin{aligned} \vec{E}^{\text{ext}}(\vec{r}) &= \vec{E}^0 + a^3 \frac{\kappa-1}{\kappa+2} \begin{pmatrix} \frac{3x^2-r^2}{r^5} & \frac{3x \cdot y}{r^5} & \frac{3x \cdot z}{r^5} \\ \frac{3y \cdot x}{r^5} & \frac{3y^2-r^2}{r^5} & \frac{3y \cdot z}{r^5} \\ \frac{3z \cdot x}{r^5} & \frac{3z \cdot y}{r^5} & \frac{3z^2-r^2}{r^5} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ E^0 \end{pmatrix}, \\ &= \begin{pmatrix} a^3 \frac{\kappa-1}{\kappa+2} \frac{3x \cdot z}{r^5} E^0 \\ a^3 \frac{\kappa-1}{\kappa+2} \frac{3y \cdot z}{r^5} E^0 \\ E^0 + a^3 \frac{\kappa-1}{\kappa+2} \frac{3z^2-r^2}{r^5} E^0 \end{pmatrix} \end{aligned} \quad (6.10)$$

The potential given by PP, eq. 5-18, is

$$\phi = a^3 \frac{\kappa-1}{\kappa+2} \frac{z}{r^3} E^0 - z E^0 \quad \text{for } r > a. \quad (6.11)$$

The field's components are therefore

$$\begin{aligned} E_x &= -\frac{\partial \phi}{\partial x} = -a^3 \frac{\kappa-1}{\kappa+2} z E^0 \frac{\partial}{\partial x} (r^{-3}) = a^3 \frac{\kappa-1}{\kappa+2} \frac{3x \cdot z}{r^5} E^0, \\ E_y &= -\frac{\partial \phi}{\partial y} = -a^3 \frac{\kappa-1}{\kappa+2} z E^0 \frac{\partial}{\partial y} (r^{-3}) = a^3 \frac{\kappa-1}{\kappa+2} \frac{3y \cdot z}{r^5} E^0, \\ E_z &= -\frac{\partial \phi}{\partial z} = E^0 - a^3 \frac{\kappa-1}{\kappa+2} E^0 \frac{\partial}{\partial z} (z r^{-3}) = E^0 + a^3 \frac{\kappa-1}{\kappa+2} \frac{3z^2-r^2}{r^5} E^0, \end{aligned} \quad (6.12)$$

coincident with the values eqs. 6.10.

The *depolarization factor* defined by por PP's eq. 5.20 (where factor ε_0 of the used edition should be in the numerator, not in the denominator)

$$L = \epsilon_0 \frac{|\vec{E}^0| - |\vec{E}^{\text{int}}|}{|\vec{P}|} = \frac{E^0 - \frac{3}{\kappa+2}E^0}{\epsilon_0 \frac{3\chi}{\kappa+2}E^0} = \frac{\kappa-1}{3(\kappa-1)} = \frac{1}{3}, \quad (6.13)$$

is the value of the three eigenvalue of the sphere's unit depolarization tensor.

Magnetic polarization

Problem 04: Deriving \mathbf{n} from the magnetic vector potential

Derive eq. for magnetic induction in terms of \mathbf{n} starting from the vector potential for uniformly magnetized matter \vec{M} :¹⁷³

$$\vec{A}(\vec{r}) = \frac{\mu_0 \lambda'}{4\pi} \iiint_V \frac{\vec{M} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3r'. \quad (6.14)$$

Solution

Usando las equations A2.2 and 3.8 se obtiene

$$\vec{A}(\vec{r}) = \frac{\mu_0 \lambda'}{4\pi} \left(\iiint_V \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r' \right) \times \vec{M} = \frac{\mu_0 \lambda'}{4\pi} \nabla I(\vec{r}) \times \vec{M}. \quad (6.15)$$

From the identity eq. A2.9,

$$\begin{aligned} \vec{B}(\vec{r}) &= \nabla \times \vec{A}(\vec{r}) = \frac{\mu_0 \lambda'}{4\pi} \nabla \times (\nabla I(\vec{r}) \times \vec{M}) \\ \nabla \times (\nabla I(\vec{r}) \times \vec{M}) &= (\vec{M} \cdot \nabla) \nabla I(\vec{r}) - (\nabla I(\vec{r}) \cdot \nabla) \vec{M} + \nabla I(\vec{r}) (\nabla \cdot \vec{M}) - \vec{M} (\nabla \cdot \nabla I(\vec{r})) \\ &= (\vec{M} \cdot \nabla) \nabla I(\vec{r}) - \vec{M} \Delta I(\vec{r}). \end{aligned} \quad (6.16)$$

Using eqs. 3.3 and A3.7 it is obtained

$$\vec{B}(\vec{r}) = \begin{cases} \mu_0 \lambda' \vec{M} - \mu_0 \lambda' \mathbf{N} \cdot \vec{M} & \text{for } \vec{r} \in V \\ -\mu_0 \lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{M} & \text{for } \vec{r} \notin V \end{cases}, \quad (6.17)$$

which are eqs. 2.35 and 2.44 for $\vec{H}^0 = 0$.

Problem 05: Permanently magnetized infinite right circular cylinder

A very long right circular cylinder of radius R has a permanent uniform magnetization M forming an angle θ_0 with the symmetry axis. Evaluate the internal

¹⁷³ Reitz, p. 191 eq. 9-12.

and external magnetic field due to the magnetization and plot the external field lines.

Solution

The internal and external magnetic fields are given by the matrix equations 2.44. The coordinate system is chosen so that the cylinder's axis corresponds to z and plane xz is parallel to vector M . From the first of eqs. 2.44,

$$\mathbf{H}^{\text{int}} = \begin{pmatrix} H_x^{\text{int}} \\ H_y^{\text{int}} \\ H_z^{\text{int}} \end{pmatrix} = -\lambda' \mathbf{N} \cdot \mathbf{M} = -\lambda' \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} M_x \\ 0 \\ M_z \end{pmatrix} = \begin{pmatrix} -\lambda' M \sin \theta_0 / 2 \\ 0 \\ 0 \end{pmatrix}. \quad (6.18)$$

That is

$$H_x^{\text{int}} = -\frac{\lambda'}{2} M \sin \theta_0, \quad H_y^{\text{int}} = 0, \quad H_z^{\text{int}} = 0, \quad H_{\perp} = -\frac{\lambda'}{2} M_{\perp}, \quad H_{\parallel} = 0, \quad (6.19)$$

where in the last two equations the components are given in the directions parallel and perpendicular to the cylinder's axis. It should be noticed that the internal magnetic field's component parallel to the cylinder's axis vanishes because of the shape anisotropy.

The external magnetic field is obtained the second of eqs. 2.44 and eq. 3.58:

$$\begin{aligned} \mathbf{H}^{\text{ext}}(\vec{r}) &= -\lambda' \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{M} \\ \begin{pmatrix} H_x^{\text{ext}} \\ H_y^{\text{ext}} \\ H_z^{\text{ext}} \end{pmatrix} &= -\frac{\lambda' R^2}{2\rho^2} \begin{pmatrix} -\cos 2\varphi & -\sin 2\varphi & 0 \\ -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} M_{\perp} \\ 0 \\ M_{\parallel} \end{pmatrix}, \\ \begin{pmatrix} H_x^{\text{ext}} \\ H_y^{\text{ext}} \\ H_z^{\text{ext}} \end{pmatrix} &= \frac{\lambda' R^2}{2\rho^2} \begin{pmatrix} M_{\perp} \cos 2\varphi \\ M_{\perp} \sin 2\varphi \\ 0 \end{pmatrix}. \end{aligned} \quad (6.20)$$

The equations of the field lines are¹⁷⁴

$$x^2 + y^2 = k \cdot y, \quad z = z_0. \quad (6.21)$$

where k and z_0 are. The geometry of these lines is shown by Figure 22. The lighter gray circumferences are the equipotential lines, function only of ρ . The k values of the field lines, with typical dipolar shape, are successive multiples of 2, except for the horizontal one for which $k = 2^{10}$. Lines are the same for any value of z .

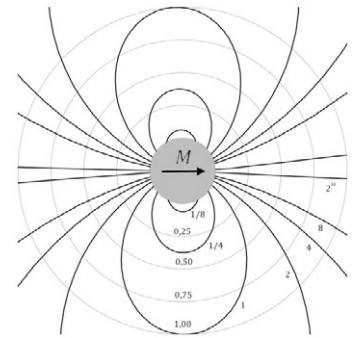


Figure 22. Field lines of a uniformly magnetized cylinder.

¹⁷⁴ Kemmer, p. 47 Problem 12.

Problem 06: Isotropic magnetic sphere

An isotropic ferromagnetic sphere is placed in a magnetic uniform field H^0 with fixed sources. Assuming the magnetization $M(H)$ to be the function given by the right graph —where M^s is the saturation magnetization and M^0 the spontaneous one— determine the resulting values of the magnetization and the magnetic field.

Solution

From eq. 2.44 it should be

$$H = H^0 - \frac{\lambda'}{3} M(H), \quad (6.22)$$

where $M(H)$ is the curve represented in Figure 23. In the same graph is represented the line

$$g(H) = \frac{H^0 - H}{\lambda'/3}, \text{ such that } g(0) = g^0 = \frac{H^0}{\lambda'/3}, \quad g(H^0) = 0. \quad (6.23)$$

The desired values of M and H are given by the intersection of that line with the function $M(H)$, because at that point

$$g(H) = \frac{H^0 - H}{\lambda'/3} = M(H), \text{ that is, } H = H^0 - \frac{\lambda'}{3} M(H). \quad (6.24)$$

The external field is the given by

$$\vec{H}(\vec{r}) = \vec{H}^0 - \frac{\lambda'}{3} \vec{n}^{\text{ext}}(\vec{r}) \vec{M}(\vec{r}), \quad (6.25)$$

where the value of \vec{n}^{ext} is that of the sphere, eq. 3.66.

Conductors

Problem 07: Solve ellipsoidal conductors using equivalent polarization

Derive the equations that solve the case of an ellipsoidal conductor in a uniform applied field using the equivalent polarization method.

Solution

Starting with the potential eq. 2.65, use the expression eq. 3.29 for the depolarization tensor assuming that the equivalent polarization is uniform. It is then obtained

$$\vec{E}(\vec{r}) = \vec{E}^0 - k_1 \nabla \oint_s \frac{\vec{P}_f \cdot d^2 \vec{r}'}{|\vec{r} - \vec{r}'|} = \vec{E}^0 - 4\pi k_1 \vec{n}(\vec{r}) \cdot \vec{P}_f. \quad (6.26)$$

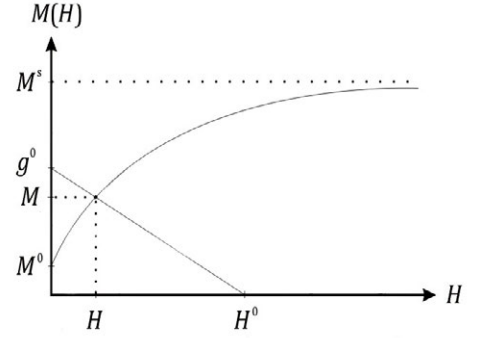


Figure 23. Graphical solution for the isotropic ferromagnetic sphere.

The equivalent polarization \vec{P}_f is determined by the vanishing internal field, and the expression of the external field is obtained in the same way as in [Problem 08](#).

Problem 08: A conductor is a perfect dielectric

Verify that ellipsoidal conductors behave as dielectrics with infinite scalar susceptibility.

Solution

Combining eqs. [2.12](#) and [2.33](#), for a dielectric with scalar χ it is obtained

$$\begin{aligned} \mathbf{E}^{\text{int}} &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{N} \cdot \mathbf{P}_{\text{eq}}, \quad \text{where} \quad \mathbf{P}_{\text{eq}} = \frac{\varepsilon_0}{\lambda} \chi (\mathbf{1} + \chi \mathbf{N})^{-1} \mathbf{E}^0 \quad \text{for } r \in V, \\ \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{P}_{\text{eq}} \quad \text{for } r \notin V. \end{aligned} \quad (6.27)$$

Taking the limit $\chi \rightarrow \infty$ in the principal system of coordinates of \mathbf{N} ,

$$\begin{aligned} \chi (\mathbf{1} + \chi \mathbf{N})^{-1} &= \begin{pmatrix} \chi (1 + \chi N_x)^{-1} & 0 & 0 \\ 0 & \chi (1 + \chi N_y)^{-1} & 0 \\ 0 & 0 & \chi (1 + \chi N_z)^{-1} \end{pmatrix}, \\ \lim_{\chi \rightarrow \infty} \chi (1 + \chi N_\alpha)^{-1} &= \lim_{\chi \rightarrow \infty} \frac{1}{\frac{1}{\chi} + N_\alpha} = N_\alpha^{-1}, \quad \lim_{\chi \rightarrow \infty} \chi (\mathbf{1} + \chi \mathbf{N})^{-1} = \mathbf{N}^{-1}. \end{aligned} \quad (6.28)$$

From the previous equation and eq. [2.66](#):

$$\begin{aligned} \mathbf{P}_{\text{eq}} &= \frac{\varepsilon_0}{\lambda} \mathbf{N}^{-1} \cdot \mathbf{E}^0, \quad \mathbf{E}^{\text{int}} = \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{N} \cdot \mathbf{P}_{\text{eq}} = 0, \\ \mathbf{E}^{\text{ext}}(\vec{r}) &= \mathbf{E}^0 - \frac{\lambda}{\varepsilon_0} \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{P}_{\text{eq}}, \quad \mathbf{p} = V \mathbf{P}_{\text{eq}} = \frac{\varepsilon_0}{\lambda} V \mathbf{N}^{-1} \cdot \mathbf{E}^0. \end{aligned} \quad (6.29)$$

These equations are the same as [2.63](#) and [2.66](#) if the following assignment is made:

$$\frac{\lambda}{\varepsilon_0} \mathbf{P}_{\text{eq}} = \mathbf{E}^+. \quad (6.30)$$

Problem 09: Polarization of a spherical conductor

Compare the charge density induced on the surface of a spherical conductor, eq. [2.57](#), that obtained by using the equivalent polarization \vec{P}_{eq} eq. [2.66](#).

Solution¹⁷⁵

From eq. 2.57, the surface charge density σ at point \vec{r}^S is

$$\sigma(\vec{r}^S) = \frac{\epsilon_0}{\lambda} \hat{s}(\vec{r}^S) \cdot \mathbf{N}^{-1} \cdot \vec{E}^0, \quad \mathbf{N} = \frac{1}{3} \mathbf{1}, \quad \hat{s}(\vec{r}^S) = \frac{\vec{r}^S}{R}. \quad (6.31)$$

From the expression of the electric dipole moment of the sphere, eq. 2.60,

$$\vec{p} = \frac{3\epsilon_0 V}{\lambda} \vec{E}^0, \quad \sigma(\vec{r}^S) = \frac{1}{V} \frac{\vec{p} \cdot \vec{r}^S}{R}. \quad (6.32)$$

The same expression is obtained from the last of eqs. 6.29 for the equivalent polarization model. This expression for σ should be compared with that given by eq. 3.32.

Problem 10: External field of a spherical conductor

Prove that the external field created by a spherical conductor immersed in a uniform electric field is that of a point dipole and find its moment¹⁷⁶.

Solution

The field of a polarized ellipsoidal conductor is given by the first two of eqs. 2.63, where the internal depolarization tensor is $\mathbf{N} = (1/3)\mathbf{1}$. That is

$$\vec{E}^{\text{ext}}(\vec{r}) = \vec{E}^0 - \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \mathbf{N}^{-1} \cdot \vec{E}^0 = (\mathbf{1} - 3\mathbf{n}^{\text{ext}}(\vec{r})) \cdot \vec{E}^0, \quad (6.33)$$

where \mathbf{n}^{ext} for a sphere is given by the second of eqs. 3.67,

$$\mathbf{n}^{\text{ext}}(\vec{r}) = -\frac{R^3}{3} \frac{3\vec{r}\vec{r} - (\vec{r} \cdot \vec{r})\mathbf{1}}{r^5}. \quad (6.34)$$

Therefore

$$\vec{E}^{\text{ext}}(\vec{r}) = \vec{E}^0 + R^3 \frac{3\vec{r}(\vec{r} \cdot \vec{E}^0) - r^2 \vec{E}^0}{r^5}. \quad (6.35)$$

The electric dipole moment of the body is given by the last two of eqs. 2.63:

$$\vec{p} = \frac{\epsilon_0 V}{\lambda} \mathbf{N}^{-1} \cdot \vec{E}^0 = \frac{4\pi\epsilon_0 R^3}{\lambda} \vec{E}^0 = \frac{1}{k_1} R^3 \vec{E}^0. \quad (6.36)$$

Upon replacement in the expression for the external electric field it is obtained

$$\vec{E}^{\text{ext}}(\vec{r}) - \vec{E}^0 = k_1 \frac{3\vec{r}(\vec{r} \cdot \vec{p}) - r^2 \vec{p}}{r^5}, \quad (6.37)$$

¹⁷⁵ Soliv  rez (2008), p. 207.

¹⁷⁶ Soliv  rez (2008), p. 207.

which is the expression for a dipolar field (see eq. A4.1).

Problem 11: Directions for which \mathbf{P} is parallel to \mathbf{E}^0 in triaxial ellipsoids

A metallic triaxial ellipsoid is placed in an arbitrary uniform field \mathbf{E}^0 . Assuming the three eigenvalues of \mathbf{N} are known, determine:

- The electric dipole moment \mathbf{p} of the ellipsoid;
- the directions along which the equivalent polarization \mathbf{P}_{eq} is parallel to the applied field \mathbf{E}^0 ;
- the angle between \mathbf{P} and \mathbf{E}^0 .

Solution

From eqs. 2.63 the electric dipole moment \mathbf{p} in the ellipsoid's principal system of coordinates —chosen so that $a > b > c$ — is

$$\mathbf{p} = \frac{\epsilon_0 V}{\lambda} \mathbf{N}^{-1} \cdot \mathbf{E}^0 = \frac{\epsilon_0 V}{\lambda} \begin{pmatrix} N_a^{-1} & 0 & 0 \\ 0 & N_b^{-1} & 0 \\ 0 & 0 & N_c^{-1} \end{pmatrix} \cdot \begin{pmatrix} E_x^0 \\ E_y^0 \\ E_z^0 \end{pmatrix}, \quad \text{where } \frac{1}{N_a} < \frac{1}{N_b} < \frac{1}{N_c}. \quad (6.38)$$

The angle δ between the equivalent polarization $\mathbf{P}_{\text{eq}} = \mathbf{p}/V$ and the field \mathbf{E}^0 is obtained from the scalar product and norm of both vectors:

$$\begin{aligned} \cos \delta &= \frac{\vec{P} \cdot \vec{E}^0}{P \cdot E^0}, \quad \vec{P} \cdot \vec{E}^0 = \sum_{\alpha} P_{\alpha} E_{\alpha}^0 = \frac{\epsilon_0}{\lambda} \sum_{\alpha} N_{\alpha}^{-1} (E_{\alpha}^0)^2, \\ P &= \frac{\epsilon_0}{\lambda} \sqrt{\sum_{\alpha} (N_{\alpha}^{-1} E_{\alpha}^0)^2}, \quad E^0 = \sqrt{\sum_{\alpha} (E_{\alpha}^0)^2}. \end{aligned} \quad (6.39)$$

Angle δ vanishes when the numerator and the denominator of the first equation are equal, which happens only along the ellipsoid's principal axes (see eq. 5.15 and Figure 18) where

$$\vec{P} \cdot \vec{E}^0 = \frac{\epsilon_0}{\lambda} N_{\alpha}^{-1} (E_{\alpha}^0)^2, \quad P = \frac{\epsilon_0}{\lambda} N_{\alpha}^{-1} E_{\alpha}^0, \quad E^0 = E_{\alpha}, \quad \alpha = x, y, z. \quad (6.40)$$

Problem 12: Electric fields generated by sharp points

Analyze the fields generated in sharp portions of conductors submitted to uniform electric fields. To that end calculate the field intensity on the intersection of the surface of a very long prolate spheroid with symmetry axis x and compare it with that on the spheroid's equator.

Solution

The field on the body's surface may be obtained from eqs. 3.41 and A7.8. For the point $(a, 0, 0)$ is obtained

$$\hat{s}(a,0,0)=\hat{x}, \quad \mathbf{n}^{\text{ext}}(a,0,0)=\begin{pmatrix} N_a-1 & 0 & 0 \\ 0 & N_b & 0 \\ 0 & 0 & N_c \end{pmatrix}. \quad (6.41)$$

From eq. 2.63 the electric field at point $(a,0,0)$ is

$$\mathbf{E}^{\text{ext}}(0,0,c)=\mathbf{E}^0-\mathbf{n}^{\text{ext}}(0,0,c)\cdot\mathbf{N}^{-1}\cdot\mathbf{E}^0, \quad (6.42)$$

where

$$\begin{aligned} & \mathbf{n}^{\text{ext}}(0,0,c)\cdot\mathbf{N}^{-1} \\ = & \begin{pmatrix} N_a-1 & 0 & 0 \\ 0 & N_b & 0 \\ 0 & 0 & N_c \end{pmatrix} \cdot \begin{pmatrix} N_a^{-1} & 0 & 0 \\ 0 & N_b^{-1} & 0 \\ 0 & 0 & N_c^{-1} \end{pmatrix} = \begin{pmatrix} 1-N_a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (6.43)$$

Therefore, the electric field has the following components normal and parallel to the x axis:

$$E_x^{\text{ext}}(0,0,c)=\frac{1}{N_z}E_x^0, \quad E_y^{\text{ext}}(0,0,c)=E_z^{\text{ext}}(0,0,c)=0. \quad (6.44)$$

When the oblate spheroid becomes very thin $a \gg b = c$ and N_a is very small. The x component of the surface field becomes very large, normal and outgoing from the body at point $(a,0,0)$. A very small applied electric field, like the ones generated during an electric storm, may thus generate a very large external field with the value given by eq. 6.44.

The equatorial electric field, tangent to the surface, vanishes as the internal one because of the continuity of the tangential component through any interphase.

Superconductors

Problem 13: Superconducting infinite cylinder and sheet

Show how to solve the infinite right circular cylinder of infinite length and the sheet of constant thickness and infinite extension in both the magnetization and surface current model. Discuss the existence of a “reduced” inverse of matrix \mathbf{N} .

Solution

Use the same method applied for a conducting infinite sheet in page 56.

Problem 14: Magnetized superconducting sphere

Evaluate the magnetization and the internal and external fields generated by a superconducting sphere under a uniform applied induction $B^0\hat{z}$. Compare with the expressions given by Reitz¹⁷⁷.

¹⁷⁷ Reitz, p. eq. 15-6.

Solution

The expressions given by Reitz (in SI units) are:

$$\vec{B} = 0, \quad \vec{H} = \frac{3}{2\mu_0} B^0 \hat{z}, \quad \vec{M} = -\frac{3}{2\mu_0} B^0 \hat{z} \quad \text{for } r \leq a. \quad (6.45)$$

$$\begin{aligned} \vec{B}(\vec{r}) = \mu_0 \vec{H}(\vec{r}) = B^0 \hat{z} - \frac{a^3}{r^3} B^0 \cos\theta \hat{r} - \frac{1}{2} \frac{a^3}{r^3} B^0 \sin\theta \hat{\theta}, \\ K_M(a, \theta, \varphi) = -\frac{3}{2\mu_0} B^0 \sin\theta \hat{\phi}. \end{aligned} \quad (6.46)$$

From eqs. 2.73, where $\mathbf{N} = (1/3)\mathbf{1}$, it is obtained in SI units ($\lambda' = 1$) for $r \leq a$:

$$\vec{B}^{\text{int}} = \mu_0 (\vec{H}^{\text{int}} + \vec{M}) = 0, \quad \vec{M} = -\frac{1}{\mu_0} \left(1 - \frac{1}{3}\right)^{-1} \vec{B}^0 = -\frac{3}{2\mu_0} \vec{B}^0 = -\vec{H}^{\text{int}}, \quad (6.47)$$

as given by Reitz.

Equation 3.66 gives \mathbf{n}^{ext} for the sphere showing that it is a dipolar field (see eq. A4.1) for $r \geq a$,

$$\begin{aligned} \vec{B}^{\text{ext}}(\vec{r}) = \vec{B}^0 - \mu_0 \mathbf{n}^{\text{ext}}(\vec{r}) \cdot \vec{M} = \vec{B}^0 - \frac{a^3}{3} \mu_0 \frac{r^2 \vec{M} - 3(\vec{M} \cdot \vec{r}) \vec{r}}{r^5} \\ = \vec{B}^0 + \frac{a^3}{2} \frac{\vec{B}^0 - 3(\vec{B}^0 \cdot \hat{r}) \hat{r}}{r^3} = \vec{B}^0 - \frac{a^3}{r^3} B^0 \cos\theta \hat{r} - \frac{1}{2} \frac{a^3}{r^3} B^0 \sin\theta \hat{\theta}, \\ \text{as } \vec{B}^0 = B^0 \hat{z} = B^0 \cos\theta \hat{r} - B^0 \sin\theta \hat{\theta}, \end{aligned} \quad (6.48)$$

as given by Reitz.

According to eqs 2.74 and A7.8 the magnetization surface current is

$$\vec{K}_M(\vec{r}) = \vec{M}(\vec{r}) \times \hat{s}(\vec{r}) = -\frac{3}{2\mu_0} \vec{B}^0 \times \hat{r} = -\frac{3}{2\mu_0} B^0 \hat{z} \times \hat{r} = -\frac{3}{2\mu_0} B^0 \sin\theta \hat{\phi}, \quad (6.49)$$

which completes the proof of perfect agreement.

Problem 15: Superconductivity as perfect diamagnetism

Verify that the equations describing superconductivity with the magnetization model may also be obtained from the perfect diamagnetic case, that is with susceptibility $\chi_m = -1$.

Solution

Induced magnetization in ellipsoids is described by eqs. 2.49. For \mathbf{H}^{int} and \mathbf{M} , where $\chi_m = -1$, it is

$$\mathbf{H}^{\text{int}} = (\mathbf{1} - \mathbf{N})^{-1} \cdot \mathbf{H}^0, \quad \mathbf{M} = \frac{1}{\lambda'} \chi_m \mathbf{H}^{\text{int}} = -\frac{1}{\lambda'} \mathbf{H}^{\text{int}}, \quad \mathbf{B}^{\text{int}} = \mu_0 (\mathbf{H}^{\text{int}} + \lambda' \mathbf{M}) = 0. \quad (6.50)$$

The external field is then give by

$$\mathbf{H}^{\text{ext}}(\vec{r}) = \mathbf{H}^0 - \lambda' \mathbf{n}(\vec{r}) \cdot \mathbf{M}. \quad (6.51)$$

The expressions are the same given by the magnetization model of superconductivity, eqs. 2.73.

Problem 16: Magnetic moment of a superconducting ellipsoid

Evaluate the magnetic dipole moment of a superconducting triaxial ellipsoid using the surface current model. Compare its value with that given by the magnetization model.

Solution

From eq. A1.12, the magnetic dipole moment of a surface distribution of current is

$$\vec{m} = \frac{k_3}{2} \oint_S \vec{r} \times \vec{K}(\vec{r}) d^2r. \quad (6.52)$$

where $\vec{K}(\vec{r})$ is current density vector whose value may be obtained from eqs. 2.75 and A1.11:

$$\vec{K}(\vec{r}) = \frac{1}{\mu_0 \lambda' k_3} \hat{s}(\vec{r}) \times \vec{B}^+, \quad (6.53)$$

It is then obtained

$$\vec{m} = \frac{1}{2\mu_0 \lambda'} \oint_S \vec{r} \times (\vec{d}^2\vec{r} \times \vec{B}^+). \quad (6.54)$$

The integrand's double vector product may be expanded as follows,

$$\vec{r} \times (\vec{d}^2\vec{r} \times \vec{B}^+) = (\vec{r} \cdot \vec{B}^+) \vec{d}^2\vec{r} - (\vec{r} \cdot \vec{d}^2\vec{r}) \vec{B}^+, \quad (6.55)$$

which replaced in the integral gives

$$\vec{m} = \frac{1}{2\mu_0 \lambda'} \oint_S (\vec{r} \cdot \vec{B}^+) \vec{d}^2\vec{r} - \frac{1}{2\mu_0 \lambda'} \left(\oint_S \vec{r} \cdot \vec{d}^2\vec{r} \right) \vec{B}^+. \quad (6.56)$$

The first surface integral may be transformed into a volume integral by using the gradient theorem eq. A3.9; the second, by using the divergence theorem eq. A3.8. Thus

$$\vec{m} = \frac{1}{2\mu_0\lambda'} \iiint_V \nabla(\vec{r} \cdot \vec{B}^+) d^3r - \frac{1}{2\mu_0\lambda'} \left(\iiint_V (\nabla \cdot \vec{r}) d^3r \right) \vec{B}^+, \quad (6.57)$$

where $\nabla(\vec{r} \cdot \vec{B}^+) = \vec{B}^+$, $\nabla \cdot \vec{r} = 3$, $\iiint_V d^3r = V$.

where use has been made of \vec{B}^+ . Using eq. 2.82 it is finally obtained

$$\vec{m} = -\frac{1}{\mu_0\lambda} V \vec{B}^+ = \frac{1}{\mu_0\lambda} \alpha_s \cdot \vec{B}^0, \quad \alpha_s = -V(\mathbf{1} - \mathbf{N})^{-1}. \quad (6.58)$$

where α_s is the body's polarizability tensor of the superconductor. This value of \vec{m} coincides with that given by eq. 2.72.

Problem 17: Description of Meissner Effect using magnetization currents

Solve the Meissner Effect in an ellipsoidal superconductor using only the surface magnetization current.

Solution

The solution follows the same steps as in the derivation surface current model of superconductivity, but using eq. 6.53.

Problem 18: Approximate solution for non ellipsoidal bodies

Find an approximate solution for the field generated by a non ellipsoidal body in one of the cases of Table 1.

Tentative solution

Use the iterative method discussed at the end of section *Solving the integro-differential equations by iteration* at page 42. This is an open problem.

Depolarization tensor

Problem 19: Components of N in spherical symmetry

Add to the symmetry operations of spheroids (see eq. 3.25) a single one that corresponds to a sphere. Use it to determine the three eigenvalues of \mathbf{N} in combination with the unit trace rule eq. 3.15. Which is the matrix representation of \mathbf{N} in a coordinate system different from the principal one?

Solution

A symmetry should be added that exchanges the spheroid's rotational axis by any of the two normal axes, for instance a rotation \mathbf{R} in 90° around coordinate axis x , as follows.

$$\begin{aligned}
\mathbf{R} \cdot \mathbf{N} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} N_{\perp} & 0 & 0 \\ 0 & N_{\perp} & 0 \\ 0 & 0 & N_{\parallel} \end{pmatrix} = \begin{pmatrix} N_{\perp} & 0 & 0 \\ 0 & 0 & N_{\parallel} \\ 0 & -N_{\perp} & 0 \end{pmatrix} \\
&= \mathbf{N} \cdot \mathbf{R} = \begin{pmatrix} N_{\perp} & 0 & 0 \\ 0 & N_{\perp} & 0 \\ 0 & 0 & N_{\parallel} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} N_{\perp} & 0 & 0 \\ 0 & 0 & N_{\perp} \\ 0 & -N_{\parallel} & 0 \end{pmatrix}.
\end{aligned} \tag{6.59}$$

from which $N_{\perp} = N_{\parallel} = N$. If the unit trace rule eq. 3.15 is applied, it follows that $3N = 1$, that is, $N = 1/3$.

As \mathbf{N} is proportional to the unit matrix, its representation is invariable in any coordinate system due to the transformation property eq. 3.23:

$$\mathbf{N}' = \mathbf{R} \cdot \left(\frac{1}{3}\mathbf{1}\right) \cdot \mathbf{R}^t = \frac{1}{3}\mathbf{1} \tag{6.60}$$

where the orthogonality relationship eq. 3.17 was used.

Problem 20: Single eigenvalue may determine the type of spheroid

An ellipsoid has an eigenvalue $N = 0.16$, the other two being equal. What is the value of the latter? To what type of ellipsoid they pertain? Which is the ratio of the smaller semiaxes to the largest?

Solution

The body is a spheroid with polar eigenvalue $N_p = 0.16$ and two equal equatorial eigenvalues N_e . From the trace rule eqs. 3.15, $N_p + 2N_e = 1$; therefore, $N_e = (1 - N_p)/2 = 0.42$, where $N_p < N_e$. As the order of semiaxes is the inverse of the order of eigenvalues, the polar semiaxes is greater then the equatorial ones, corresponding to a prolate spheroid (see Figure 20). In the conventional order $a \geq b \geq c$, $N_a = 0.16$, $N_b = N_c = 0.42$. The ratio of semiaxes $\beta = \gamma$ may be obtained from eq. 4.30 or from Figure 14; the latter being enough for the given precision. By inspection it is easily seen that $\beta = \gamma \cong 0.47$, value that may be checked using Figure 15 for $N_b = N_c = 0.42$ and adjusted for higher precision using eq. 4.30.

Problem 21: Eigenvalues and aspect ratios

Using each of the following set of data, identify all the eigenvalues of \mathbf{N} and the aspect ratios of the ellipsoid to which it pertain.

- $N = 0.65$; $\gamma = 0.10$.
- $N = 0.60$; $\beta = 0.50$.
- $N_1 = N_2 = 0.40$.

Solutions

- The range of values 0.5-1.0 pertains only to N_c . As seen from Figure 16, for $\gamma = 0.10$ the value $N_c = 0.65$ is on the curve $\beta = 0.20$. The other eigenvalues for this aspect ratio are $N_b = 0.32$ (obtained from Figure 15) and $N_a = 0.03$ (Figure

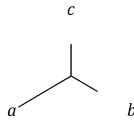
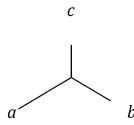
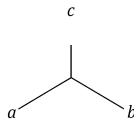
14).

- b) A similar thing happens in this case, where $N_c = 0.60$ and $\beta = 0.50$, where from Figure 16 it is found $\gamma = 0.25$. The two remaining eigenvalues are $N_b = 0.28$ (obtained from Figure 15) and $N_a = 0.11$ (Figure 14).
- c) The only possible spheroid is the prolate one with $N_b = N_c = 0.40$, $\beta = \gamma = 0.57$ and $N_a = 0.20$.

Problem 22: Sequence of ellipsoids over some aspect ratio curves

Over the line $\gamma = 0.5$, analyze the changes of shape of the ellipsoid corresponding to N_a , N_b and N_c at three different values of β . Devise a simple geometric representation that shows clearly the changes in the ratio of semiaxes.

Solution

β	γ	N_a	N_b	N_c	Ellipsoid	
0.50	0.50	0.17	0.41	0.41	prolate spheroid	
0.75	0.50	0.21	0.31	0.49	triaxial	
1.00	0.50	0.24	0.24	0.51	oblate spheroid	

Problem 23: Eigenvalues of triaxial ellipsoids

Using the data given by Osborn and Stoner calculate the principal values of \mathbf{N} for an ellipsoid such that its semiaxes satisfy the relationships 1 : 2 : 3.

Solution

The values can be obtained from the tables and graphs of Osborne and Stoner. As both use the convention $a \geq b \geq c$, we may choose $a = 3$, $b = 2$, $c = 1$, with aspect ratios $\gamma = c/a = 0.33333$; $\beta = b/a = 0.66666$.

From Osborne's Table I the following approximate values are obtained:

$$\begin{aligned}
 &c/a = 0.34202 \text{ and } b/a = 0.69414: \\
 &N_a = 0.16176; N_b = 0.26200; N_c = 0.57624. \\
 &c/a = 0.34202 \text{ and } b/a = 0.58115: \\
 &N_a = 0.14940; N_b = 0.30440; N_c = 0.54620.
 \end{aligned}
 \tag{6.61}$$

From Stoner's graph, that gives larger errors, one obtains $N_a = 0.15$; $N_b = 0.28$; $N_c = 0.55$.

Problem 24: Depolarization tensor of a very long right circular cylinder

Compute the depolarization tensor, eq. 3.4, of a right circular cylinder of infinite length using electrostatic Gauss's Law.

Solution

See section *Right circular cylinder of infinite length* of chapter Chapter 3.

Problem 25: Depolarization tensor of a sphere

Compute the depolarization tensor of a sphere, eq. 3.4, using electrostatic Gauss's Law.

Solution

See section *Sphere* of chapter 3.

Problem 26: Alternative way of obtaining certain eigenvalues of N

Explore alternatives ways of obtaining the eigenvalues of **N** not given in this book. See, for instance, MacMillan's equations 32.10 and 39.4. Bear in mind the discussion made at section *Unified treatment of N for spheroids*. Test your approach deriving the depolarization factors of the two types of spheroids.

Solution

Open problem.

Problem 27: Solving the prolate spheroid with Legendre's elliptic integrals

Show that the principal values for the prolate spheroids may be obtained from the general expression in terms of normal of elliptic integrals eq. 4.9.

Solution

The prolate spheroid corresponds to the case $k = 1$, for which E and F are given by eq. A9.1, and ϕ should be expressed in terms of $\beta = \gamma$.

$$E(\phi, 1) = \sin \phi, \quad F(\phi, 1) = \int_0^\phi \frac{dt}{\cos t} = \ln \left(\tan \left(\phi / 2 + \pi / 4 \right) \right), \quad \cos \phi = \gamma, \quad (6.62)$$

where use has been made of the last of eqs. 4.9. Expressing the tangent in terms of cosine, it is obtained

$$\begin{aligned} \tan \left(\phi / 2 + \pi / 4 \right) &= \frac{\sin \left(\phi / 2 + \pi / 4 \right)}{\cos \left(\phi / 2 + \pi / 4 \right)} \\ &= \frac{\sin(\phi / 2) \cos(\pi / 4) + \cos(\phi / 2) \sin(\pi / 4)}{\cos(\phi / 2) \cos(\pi / 4) - \sin(\phi / 2) \sin(\pi / 4)} = \frac{\sin(\phi / 2) + \cos(\phi / 2)}{\cos(\phi / 2) - \sin(\phi / 2)}, \end{aligned} \quad (6.63)$$

as $\sin(\pi/4) = \cos(\pi/4)$. Taking into account the following trigonometric identity¹⁷⁸,

¹⁷⁸ Korn & Korn, p. 810 eq. 21.2.9.

$$\sin(\phi/2) = \sqrt{\frac{1-\cos\phi}{2}}, \quad \cos(\phi/2) = \sqrt{\frac{1+\cos\phi}{2}} \quad \text{for } 0 < \phi < \frac{\pi}{2}, \quad (6.64)$$

it is finally obtained

$$\begin{aligned} \tan(\phi/2 + \pi/4) &= \frac{\sin(\phi/2) + \cos(\phi/2)}{\cos(\phi/2) - \sin(\phi/2)} \\ &= \frac{\left(\sqrt{\frac{1-\cos\phi}{2}} + \sqrt{\frac{1+\cos\phi}{2}}\right) \left(\sqrt{\frac{1+\cos\phi}{2}} + \sqrt{\frac{1-\cos\phi}{2}}\right)}{\left(\sqrt{\frac{1+\cos\phi}{2}} - \sqrt{\frac{1-\cos\phi}{2}}\right) \left(\sqrt{\frac{1+\cos\phi}{2}} + \sqrt{\frac{1-\cos\phi}{2}}\right)} \\ &= \frac{\left(\sqrt{1+\cos\phi} + \sqrt{1-\cos\phi}\right)^2}{1+\cos\phi - (1-\cos\phi)} = \frac{1+\sqrt{1-\cos^2\phi}}{\cos\phi} = \frac{1+\sqrt{1-\gamma^2}}{\gamma}. \end{aligned} \quad (6.65)$$

Therefore

$$E(\phi, 1) = \sqrt{1-\gamma^2}, \quad F(\phi, 1) = \ln\left(\frac{1+\sqrt{1-\gamma^2}}{\gamma}\right). \quad (6.66)$$

As $\beta = \gamma$ replacement of this values in the first of eqs. 4.9 gives

$$\begin{aligned} N_x &= -\frac{\beta \cdot \gamma \cdot E(\phi, 1)}{(1-\beta^2)\sqrt{1-\gamma^2}} + \frac{\beta \cdot \gamma}{(1-\beta^2)\sqrt{1-\gamma^2}} F(\phi, 1) \\ &= \frac{\beta^2}{1-\beta^2} \left(\frac{1}{\sqrt{1-\beta^2}} \ln\left(\frac{1+\sqrt{1-\beta^2}}{\beta}\right) - 1 \right). \end{aligned} \quad (6.67)$$

The value coincides with that of eq. 4.30, obtained by direct integration of eq. 3.82, which serves at the same time of verification of the expression in normal elliptic integrals eq. A9.1.

$$\begin{aligned} N_y &= -\frac{\gamma^2}{\beta^2 - \gamma^2} + \frac{\beta \cdot \gamma \cdot (1-\gamma^2)}{(\beta^2 - \gamma^2)(1-\beta^2)} - \frac{\beta \cdot \gamma}{(1-\beta^2)\sqrt{1-\gamma^2}} \ln\left(\frac{1+\sqrt{1-\gamma^2}}{\gamma}\right), \\ N_z &= \frac{\beta^2}{(\beta^2 - \gamma^2)} - \frac{\beta \cdot \gamma \cdot \sqrt{1-\gamma^2}}{(\beta^2 - \gamma^2)\sqrt{1-\gamma^2}}. \end{aligned} \quad (6.68)$$

Problem 28: Spherical shell

Explore the existence of a depolarization tensor for a spherical shell (homoeoid), using Gauss's Law.

Solution

Gauss's theorem will be used to find the potential of the homoeoid of Figure 24 where the gray region is homogeneously charged matter. The "depolarization

tensor" will then be calculated by derivation of the potential. The shell is assumed to have constant volume density of charge ρ . For the calculation of the flux of the electric field the surfaces S_1 , S_2 and S_3 will be used, corresponding to the three regions $r < R_1$, $R_1 < r < R_2$ and $R_2 < r$. From the symmetry of the problem the electric field has only radial component $E(r)$.

Over S_1 the flux is

$$\oiint_{S_1} \vec{E} \cdot d\vec{S} = 4\pi r^2 E = 4\pi k_1 Q_1 = 0, \quad E = 0. \quad (5.69)$$

Over S_2 it is

$$\begin{aligned} \oiint_{S_2} \vec{E} \cdot d\vec{S} &= 4\pi r^2 E = 4\pi k_1 Q_2 \\ &= 4\pi k_1 \rho \frac{4\pi}{3} (r^3 - R_1^3), \\ E &= \frac{4\pi k_1 \rho}{3} \left(r - \frac{R_1^3}{r^2} \right) \quad \text{for } R_1 \leq r \leq R_2. \end{aligned} \quad (6.70)$$

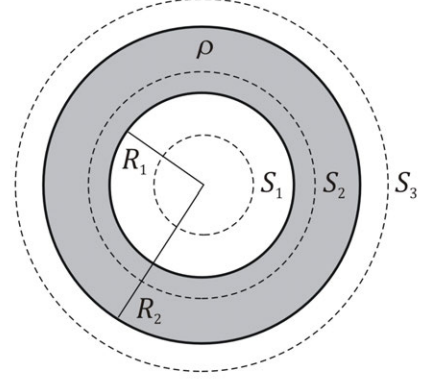


Figure 24. Spherical homoeoid.

Over S_3 it is obtained

$$\begin{aligned} \oiint_{S_3} \vec{E} \cdot d\vec{S} &= 4\pi r^2 E = 4\pi k_1 Q_3 = 4\pi k_1 \rho \frac{4\pi}{3} (R_2^3 - R_1^3), \\ E &= \frac{4\pi k_1 \rho}{3} \left(\frac{R_2^3 - R_1^3}{r^2} \right) \quad \text{for } R_2 \leq r. \end{aligned} \quad (6.71)$$

The electric potential ϕ is obtained by integration over r of the previous expressions, from which it is obtained f (see eq. 3.43):

$$f(\vec{r}) = -\frac{\phi(r)}{4\pi k_1 \rho} = \frac{1}{4\pi k_1 \rho} \int E(r) dr = \begin{cases} \text{const. for } r \leq R_1, \\ \text{const.} + \frac{1}{6} r^2 + \frac{R_1^3}{3} r^{-1} & \text{for } R_1 \leq r \leq R_2, \\ \text{const.} + \frac{R_2^3 - R_1^3}{3} r^{-1} & \text{for } R_2 \leq r. \end{cases} \quad (6.72)$$

values that coincide with those given by MacMillan¹⁷⁹. The potential is the difference of the potentials of the spheres of radius R_2 and R_1 . The double derivatives of f (see eq. 3.8) do not define a valid (constant) internal depolarization tensor.

¹⁷⁹ MacMillan, p. 40.

Problem 29. Unit vector normal to the ellipsoid's surface

Verify that vector \hat{s} , eq. A7.8, is normal to the surface at the intersections with the three cartesian axes and at every point of the ellipses determined by the intersection with the planes defined by an arbitrary pair of coordinate axes.

Solution

Unit vector $\hat{s}(\vec{r})$ is given by

$$\hat{s}(\vec{r}) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \left(\frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z} \right). \quad (6.73)$$

At the intersection with any coordinate axis the ellipsoidal surface is trivially normal to it. Less trivial is the condition for the elliptic section determined by a pair of coordinate axes, like the plane xy discussed below. The corresponding ellipsoid's central section for $z = 0$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (6.74)$$

The equation of the straight line normal to the ellipse at point (x_1, y_1) is¹⁸⁰

$$\frac{y - y_1}{x - x_1} = \frac{a^2 y_1}{b^2 x_1}. \quad (6.75)$$

Vector \hat{s} at that point is

$$\hat{s}(\vec{r}_1) = \left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} \right)^{-\frac{1}{2}} \left(\frac{x_1}{a^2} \hat{x} + \frac{y_1}{b^2} \hat{y} \right), \quad (6.76)$$

whose components have the ratio

$$\frac{s_y(\vec{r}_1)}{s_x(\vec{r}_1)} = \frac{y_1^2 a^2}{b^2 x_1^2}, \quad (6.77)$$

coincident with the slope of the straight line normal to the ellipse at that point.

Problem 30: Surface step discontinuity of the depolarization tensor

Prove eq. 3.38 without making reference to an electric or magnetic fields.

Solution

Hint: use eq. 3.29.

¹⁸⁰ Korn and Korn, p. 46.

Problem 31: Ellipsoidal conductor with net charge

Discuss the feasibility of using the depolarization tensor method for obtaining the surface charge distribution on ellipsoidal conductors with net charge in an applied electric field.

Solution

It is an open problem not solved by the author. The depolarization tensor method seems to require zero net charges.

Problem 32: Application of Ivory's method to the sphere

As an illustration of Ivory's method given in section *Obtention of* the external gravitational potential by Ivory's method, find the internal and external potential of a homogeneous spherical mass.

Solution

Internal value

The potential inside the body is given by eqs. 4.4 and 4.5, so it is necessary to evaluate the following integrals.

$$\int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = \int_0^\infty \frac{ds}{(R^2+s)^{3/2}} = -2(R^2+s)^{-1/2} \Big|_0^\infty = \frac{2}{R}, \quad (6.78)$$

$$\int_0^\infty \frac{ds}{(d_\alpha^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = \int_0^\infty \frac{ds}{(R^2+s)^{5/2}} = -\frac{2}{3}(R^2+s)^{-3/2} \Big|_0^\infty = \frac{2}{3} \frac{1}{R^3}. \quad (6.79)$$

The potential eq. 4.2 is therefore

$$\phi(\vec{r}) = C_0(\lambda) + C_x(\lambda)x^2 + C_y(\lambda)y^2 + C_z(\lambda)z^2 = M \left(\frac{1}{2R} - \frac{1}{2R^3} r^2 \right), \quad (6.80)$$

where M is the mass of the sphere. The expression, *mutatis mutandis*, coincides with the potential of a uniformly charged sphere inside the body (eq. 3.62).

External value

The value of κ is determined by solving eq. A7.20, that for the sphere of radius R becomes

$$\frac{x^2 + y^2 + z^2}{R^2 + \kappa} = \frac{r^2}{R^2 + \kappa} = 1, \quad \text{giving } \kappa = r^2 - R^2. \quad (6.81)$$

This value should be used in eqs. 4.52 and 4.54. The integrals to evaluate are

$$\int_\kappa^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = \int_{r^2-R^2}^\infty \frac{ds}{(R^2+s)^{3/2}} = -2(s+R^2)^{-1/2} \Big|_{r^2-R^2}^\infty = \frac{2}{r}, \quad (6.82)$$

$$\int_{\kappa}^{\infty} \frac{ds}{(d_{\alpha}^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} = \int_{r^2 - R^2}^{\infty} \frac{ds}{(R^2 + s)^{5/2}} = -\frac{2}{3} (R^2 + s)^{-3/2} \Big|_{r^2 - R^2}^{\infty} = \frac{2}{3} \frac{1}{r^3}. \quad (6.83)$$

The potential eq. 4.2 at points external to the body is therefore

$$\begin{aligned} \phi(\vec{r}) &= C_0(\lambda) + C_x(\lambda)x^2 + C_y(\lambda)y^2 + C_z(\lambda)z^2 \\ &= \pi\sigma abc \left(2\frac{1}{r} - \frac{2}{3} \frac{x^2 + y^2 + z^2}{r^3} \right) = \frac{4\pi R^3 \sigma}{3} \frac{1}{r} = \frac{M}{4} \frac{1}{r}, \end{aligned} \quad (6.84)$$

where M is the mass of the sphere. The expression, *mutatis mutandis*, coincides with the potential of a uniformly charged sphere outside the body (eq. 3.62).

Problem 33: Expression for far away fields obtained from n^{ext}

Using Ivory's method give the expression of n^{ext} at large distances from a uniformly polarized triaxial ellipsoid.

Solution

Far away from the body the radial distance $r \gg a$ and $\kappa \gg a^2$. Therefore, in the equation A7.20 that determines κ it is valid to disregard the squared semiaxes in the denominator so that

$$\frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} \cong \frac{r^2}{\kappa} \cong 1, \quad \kappa \cong r^2. \quad (6.85)$$

These value of κ should now be used in the expression for $n_{\alpha\beta}^{\text{ext}}(\vec{r})$, eqs. 4.66, with

$$\begin{aligned} N_{\alpha}(\kappa) &= \frac{abc}{2} \int_{\kappa}^{\infty} \frac{ds}{(d_{\alpha}^2 + s) \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}} \\ &= \frac{abc}{2} \int_{r^2}^{\infty} \frac{ds}{s^{5/2}} = -\frac{abc}{3} s^{-3/2} \Big|_{r^2}^{\infty} = \frac{abc}{3} \frac{1}{r^3}, \end{aligned} \quad (6.86)$$

$$\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)} = r^3, \quad (6.87)$$

$$s_{\alpha}(\vec{r} | \kappa) = \frac{\frac{x_{\alpha}}{d_{\alpha}^2 + \kappa}}{\sqrt{\frac{x^2}{(a^2 + \kappa)^2} + \frac{y^2}{(b^2 + \kappa)^2} + \frac{z^2}{(c^2 + \kappa)^2}}} = \frac{x_{\alpha}}{r}. \quad (6.88)$$

Therefore

$$\begin{aligned}\mathbf{n}^{\text{ext}}(\vec{r}) &= \mathbf{N}(\kappa) - \frac{abc}{\sqrt{(a^2 + \kappa)(b^2 + \kappa)(c^2 + \kappa)}} \vec{s}(\vec{r}|\kappa) \vec{s}(\vec{r}|\kappa) \\ &= \frac{1}{4\pi} \frac{V}{3} \left(\frac{r^2}{r^5} \mathbf{1} - 3 \frac{\vec{r}\vec{r}}{r^5} \right),\end{aligned}\quad (6.89)$$

where V is the volume of the ellipsoid eq. A7.4. The tensor, equal to that of the sphere eq. 3.66, corresponds to a dipolar field where the dipole moment is V times the electric or magnetic polarization (see eq. 6.36).

Problem 34: Field on the external side of the surface of a polarized sphere

Verify the validity of eq. 3.41 giving the value of \mathbf{n}^{ext} on the surface of a sphere.

Solution

The quoted expression is

$$\mathbf{n}^{\text{ext}}(\vec{r}^S) = \mathbf{N} - \hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S), \quad (6.90)$$

where, as for the sphere $a = b = c = R$,

$$\hat{s}(\vec{r}^S) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \left(\frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z} \right) = \frac{\vec{r}^S}{R}, \quad \text{where } R = |\vec{r}^S|. \quad (6.91)$$

From eqs. 3.66, \mathbf{N} and the surface value of the external depolarization tensor for the sphere of radius R are given by

$$\mathbf{N} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \mathbf{n}^{\text{ext}}(\vec{r}^S) = \begin{pmatrix} \frac{1}{3} - \frac{x^2}{R^2} & -\frac{x \cdot y}{R^2} & -\frac{x \cdot z}{R^2} \\ -\frac{y \cdot x}{R^2} & \frac{1}{3} - \frac{y^2}{R^2} & -\frac{y \cdot z}{R^2} \\ -\frac{z \cdot x}{R^2} & -\frac{z \cdot y}{R^2} & \frac{1}{3} - \frac{z^2}{R^2} \end{pmatrix}. \quad (6.92)$$

In matrix notation $\hat{s}(\vec{r}) \hat{s}(\vec{r})$ is

$$\hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S) = \frac{1}{R^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} x & y & z \end{pmatrix} = \frac{1}{R^2} \begin{pmatrix} x^2 & x \cdot y & x \cdot z \\ y \cdot x & y^2 & y \cdot z \\ z \cdot x & z \cdot y & z^2 \end{pmatrix}. \quad (6.93)$$

Therefore, the two values coincide and

$$\mathbf{n}^{\text{ext}}(\vec{r}^S) = \mathbf{N} - \hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S). \quad (6.94)$$

Problem 35: Microscopic origin of depolarization tensors

Discuss the derivation of depolarization tensors from microscopic magnitudes such as the tensor \mathbf{m} discussed at page 24.

Solution

Apart from the argument given at page 24, it is an open problem of interest for the discussion of the relationship between microscopic and macroscopic electromagnetic fields.

Energy, forces and torques

Problem 36: Energy of a dielectric sphere

A dielectric sphere of radius R is uniformly polarized by a uniform applied field \vec{E}^0 . Verify, integrating $\epsilon_0 \vec{E}^p \cdot \vec{E}^p$ over all space, that there is no contribution to the total free energy from the depolarization field \vec{E}^p .

Solution

The external field \vec{E}_p^{ext} generated by the sphere's polarization \vec{P} is the dipolar one eqs. 6.10, A4.1 and 6.8:

$$\vec{E}_p^{\text{ext}}(\vec{r}) = k_1 \frac{3(\vec{p} \cdot \vec{r})\vec{r} - r^2 \vec{p}}{r^5}, \quad \text{where} \quad \vec{p} = V \vec{P} = V \frac{3\epsilon_0(\kappa - 1)}{\kappa + 2} \vec{E}^0. \quad (6.95)$$

The contribution to the energy of this external field is (see eqs. A4.11 and A1.4):

$$F_e^{\text{ext}} = \frac{\epsilon_0}{2\lambda} \iiint_{V'} \vec{E}_p^{\text{ext}}(\vec{r}) \cdot \vec{E}_p^{\text{ext}}(\vec{r}) d^3r = \frac{\epsilon_0}{\lambda} \left(\frac{4\pi k_1}{3} \right)^2 \frac{p^2}{V} = \frac{1}{9} \frac{\lambda}{\epsilon_0} V \vec{P} \cdot \vec{P} \quad (6.96)$$

The contribution to the energy of the internal depolarization field \vec{E}_p^{int} is (eqs. 2.12 and 3.66)

$$\begin{aligned} \vec{E}_p^{\text{int}} &= -\frac{\lambda}{\epsilon_0} \mathbf{N} \cdot \vec{P} = -\frac{\lambda}{3\epsilon_0} \vec{P}, \quad \vec{D}_p^{\text{int}} = \epsilon_0 \vec{E}_p^{\text{int}} + \lambda \vec{P} = \frac{2}{3} \lambda \vec{P}, \\ F_e^{\text{int}} &= \frac{1}{2\lambda} \iiint_V \vec{E}_p^{\text{int}} \cdot \vec{D}_p^{\text{int}} d^3r = \frac{V}{2\lambda} (\vec{E}_p^{\text{int}} \cdot \vec{D}_p^{\text{int}}) = -\frac{1}{9} \frac{\lambda}{\epsilon_0} V \vec{P} \cdot \vec{P}, \end{aligned} \quad (6.97)$$

which cancels the external contribution.

Problem 37: Fermi's contact interaction

For an atom, find the energy of interaction between the electronic current density and the magnetization of the nucleus. Approximate the latter by a uniformly magnetized triaxial ellipsoid. Compare with the standard expression of Fermi's contact term U_F and discuss the origin of the difference.

$$U_F = -\frac{1}{3}2\lambda\mu_0\vec{\mu}_n \cdot \vec{m}_e, \quad (6.98)$$

where $\vec{\mu}_n$ is the nuclear magnetic moment and \vec{m}_e the electronic magnetization over the nucleus¹⁸¹.

Problem 38: Torques on spheroids

Compute the torque exerted over an isotropic dielectric spheroid immersed in a uniform applied field. Is it possible to obtain the value of the dielectric susceptibility only from this value?

Solution

The force couple is given by eqs. 5.21 that, in the spheroid's principal system with z as the symmetry axis, gives

$$\begin{aligned} \tau_x &= \frac{\epsilon_0}{\lambda} V \chi_e^2 \left(\frac{N_{\parallel} - N_{\perp}}{(1 + \chi_e N_{\parallel})(1 + \chi_e N_{\perp})} \right) E_y^0 E_z^0, \\ \tau_y &= -\frac{\epsilon_0}{\lambda} V \chi_e^2 \left(\frac{N_{\parallel} - N_{\perp}}{(1 + \chi_e N_{\parallel})(1 + \chi_e N_{\perp})} \right) E_z^0 E_x^0, \\ \tau_z &= \frac{\epsilon_0}{\lambda} V \chi_e^2 \left(\frac{N_{\perp} - N_{\parallel}}{(1 + \chi_e N_{\perp})^2} \right) E_x^0 E_y^0 = 0. \end{aligned} \quad (6.99)$$

The expression of the field components in spherical coordinates is

$$\begin{aligned} E_x^0 &= E^0 \sin\theta \cos\varphi, \quad E_y^0 = E^0 \sin\theta \sin\varphi, \quad E_z^0 = E^0 \cos\theta, \\ \tau_x &= k(E^0)^2 \sin\theta \cos\theta \sin\varphi = \frac{k}{2}(E^0)^2 \sin 2\theta \sin\varphi, \\ \tau_y &= -k(E^0)^2 \sin\theta \cos\theta \cos\varphi = -\frac{k}{2}(E^0)^2 \sin 2\theta \cos\varphi, \\ \vec{\tau} &= \frac{k}{2}(E^0)^2 \sin 2\theta (\hat{x} \sin\varphi - \hat{y} \cos\varphi) = \frac{k}{2}(E^0)^2 \sin 2\theta \left(\hat{x} \cos(\varphi - \frac{\pi}{2}) + \hat{y} \sin(\varphi - \frac{\pi}{2}) \right), \\ k &= \frac{\epsilon_0}{\lambda} V \chi_e^2 \left(\frac{N_{\parallel} - N_{\perp}}{(1 + \chi_e N_{\parallel})(1 + \chi_e N_{\perp})} \right). \end{aligned} \quad (6.100)$$

The torque is normal to the plane determined by the symmetry axis and the field, and tends to turn the largest semiaxis towards the field. The value of k can be

¹⁸¹ C. E. Soliv  rez; *The contact hyperfine interaction: an ill defined problem*; J. Phys. Solid St. Phys. vol. 13, L1017-L1019; 1980. Although the problem is of quantum nature, the discussion is made in terms of classical electromagnetism.

determined from the amplitude of the function $\tau(\theta)$. As all other parameters are known, χ_e may then be found by solving a second degree equation.

Problem 39: Torque exerted over an anisotropic dielectric sphere

A spherical and anisotropic dielectric is suspended in a uniform and fixed electric field. Express the torque exerted over the sphere in terms of the principal values of the electric susceptibility tensor.

Solution

For a sphere of anisotropic uniaxial dielectric (see Table 5) it is sufficient to study the force couple on any plane containing the symmetry axis, using as coordinates the principal system of the susceptibility tensor. z axis is taken to be the symmetry of revolution one and plane yz as containing the applied field \vec{E}^0 . From eqs. 5.21 and 3.66 it is then obtained

$$\begin{aligned}\vec{\tau} &= \vec{p} \times \vec{E}^0, \quad \vec{p} = \frac{\epsilon_0}{\lambda} \alpha_e \cdot \vec{E}^0, \quad \alpha_e = V \chi_e \cdot \left(1 + \frac{1}{3} \chi_e\right)^{-1}, \quad \vec{E}^0 = E_y^0 \hat{y} + E_z^0 \hat{z}, \\ \vec{p} &= \frac{\epsilon_0}{\lambda} V \left(\frac{\chi_{\perp}}{1 + \frac{1}{3} \chi_{\perp}} (\hat{x}\hat{x} + \hat{y}\hat{y}) + \frac{\chi_{\parallel}}{1 + \frac{1}{3} \chi_{\parallel}} \hat{z}\hat{z} \right) \cdot (E_y^0 \hat{y} + E_z^0 \hat{z}) \\ &= \frac{\epsilon_0}{\lambda} V \left(\frac{\chi_{\perp} E_y^0}{1 + \frac{1}{3} \chi_{\perp}} \hat{y} + \frac{\chi_{\parallel} E_z^0}{1 + \frac{1}{3} \chi_{\parallel}} \hat{z} \right).\end{aligned}\tag{6.101}$$

The force couple is then

$$\begin{aligned}\vec{\tau} &= \frac{\epsilon_0}{\lambda} V \left(\frac{\chi_{\perp} E_y^0}{1 + \frac{1}{3} \chi_{\perp}} \hat{y} + \frac{\chi_{\parallel} E_z^0}{1 + \frac{1}{3} \chi_{\parallel}} \hat{z} \right) \times (E_y^0 \hat{y} + E_z^0 \hat{z}) \\ &= \frac{\epsilon_0}{\lambda} V \left(\frac{\chi_{\perp}}{1 + \frac{1}{3} \chi_{\perp}} - \frac{\chi_{\parallel}}{1 + \frac{1}{3} \chi_{\parallel}} \right) E_y^0 E_z^0 \hat{x} = \frac{\epsilon_0}{3\lambda} V \left(\frac{\chi_{\parallel} - \chi_{\perp}}{\left(1 + \frac{1}{3} \chi_{\perp}\right) \left(1 + \frac{1}{3} \chi_{\parallel}\right)} \right) E_y^0 E_z^0 \hat{x}.\end{aligned}\tag{6.102}$$

For an applied field in quadrant xy, the exerted torque makes the body align its smallest susceptibility eigenvalue with the applied field.

Problem 40: Torque on a magnetized disk

A constant thickness circular disk of Fe-Si alloy is permanently magnetized in a uniform way. The disk is suspended from its edge in a uniform and horizontal magnetic field. Determine the value of its magnetization from the torque exerted by the field on the disk. See the experimental configuration at Figure 21.

Solution

A circular disk of thickness much smaller than its diameter may be approximated by an oblate spheroid of polar semiaxis equal to half its thickness and an equatorial semiaxis equal to its radius. From the discussion at page 111 about magnetization of a compass's needle, the magnetization vector should lie on the equatorial plane.

If the disk can rotate freely, The magnetization will align with the field so that the free energy eq. 5.35 has it minimum value.

The coordinate system is chosen so that the suspension is parallel to the z axis, the magnetization lies on the plane xy and the field lies along the direction x . Then

$$\vec{\tau}(\varphi) = \vec{m} \times \vec{B}^0 = VM(\cos\varphi \hat{x} + \sin\varphi \hat{y}) \times B^0 \hat{x} = -VMB^0 \sin\varphi \hat{z}. \quad (6.103)$$

The value of M is obtained from the amplitude of the torque as a function of φ .

Problem 41: Torque on a very long superconducting right circular cylinder

Compute the torque exerted over a very long cylindrical superconductor suspended normally to its symmetry axis in horizontal and uniform magnetic field.

Solution

The torque over a body of large volume is very large, and large bodies are very difficult to keep at the temperature where they are superconductors. The problem is unrealistic and without practical interest.

Problem 42: Depolarization tensor of infinitesimally thin shells

Verify that all infinitesimally thin shells of ellipsoidal shape have a depolarization tensor similar to that of the spherical one of eq. 5.56:

$$\mathbf{N} = \hat{s}(\vec{r}^S) \hat{s}(\vec{r}^S), \quad \mathbf{n}^{\text{ext}} = 0. \quad (6.104)$$

where $\hat{s}(\vec{r}^S)$ is the unit vector normal at point \vec{r}^S of the body's surface (see eq. A7.8).

Solution

Use the superposition principle for the potential of ellipsoids with constant density of charge.

Problem 43: Applications of thin shells

Explore the application of thin ellipsoidal shells for solving a practical problem.

Solution

Open answer.

Appendix 1: Electromagnetic units

The different systems of electromagnetic units are characterized by the constants appearing in the basic laws of electromagnetism. While almost without exception engineers use the International System of Units¹⁸² (SI), physicists often use other systems more convenient for their particular field of studies. For that reason these general constants are used in this book, following Jackson's ideas¹⁸³. A few other constants are next discussed that depend on the definition of certain derived but relevant magnitudes, as electric and magnetic susceptibilities and moments.

Coulomb's Law

$$F = k_1 \frac{qq'}{r^2}, \quad (\text{A1.1})$$

defines in some systems the unit of electric charge, through k_1 . The constant in **Gauss's Law** is a consequence of this definition,

$$\oint_S \vec{E} \cdot d\vec{S} = 4\pi k_1 Q. \quad (\text{A1.2})$$

Electric displacement is defined in terms of electric field and electric polarization density,

$$\vec{D} = \epsilon_0 \vec{E} + \lambda \vec{P}, \quad (\text{A1.3})$$

where ϵ_0 is vacuum permittivity. λ is an adimensional constant that makes electric displacement a magnitude of the same species than electric polarization.

For all the systems of units here discussed it happens that

$$\frac{4\pi k_1 \epsilon_0}{\lambda} = 1. \quad (\text{A1.4})$$

The **electric dipole moment** p of two point charges $+q$, $-q$ at distance d is

¹⁸² [Electromagnetic units](#) at the Bureau International des Poids et Mesures (BIPM), SI maintenance agency.

¹⁸³ Jackson, pp. 611-621. The book does not analyze the relationships between E and B , P and M brought by the theory of relativity, which surely establishes more relationships between constants, perhaps those given by eqs. [A1.4](#) and [A1.10](#).

$$p = \frac{4\pi k_1 \varepsilon_0}{\lambda} q \cdot d = q \cdot d, \quad (\text{A1.5})$$

where eq. A1.4 was used.

The unit of **electric current** I is defined by the force per unit length exerted on two parallel conductors a distance d apart:

$$\frac{\Delta F}{\Delta l} = 2k_2 \frac{I \cdot I'}{d}. \quad (\text{A1.6})$$

From Maxwell's equations A1.13, constants k_1 and k_2 are related by the **velocity of light** c , that of propagation of electromagnetic waves in vacuum,

$$\frac{k_1}{k_2} = c^2. \quad (\text{A1.7})$$

The unit of magnetic induction is defined by the magnetic part of the **Lorentz force** exerted over a charge q moving at velocity \vec{v} :

$$\vec{F} = k_3 q \vec{v} \times \vec{B}. \quad (\text{A1.8})$$

Amp  re's Law is the magnetic analogue of Gauss's Law,

$$\oint_c \vec{B} \cdot d\vec{l} = \frac{4\pi k_2}{k_3} I. \quad (\text{A1.9})$$

The relationship between magnetic induction \vec{B} , magnetic field and magnetization is

$$\vec{B} = \mu_0 (\vec{H} + \lambda' \vec{M}), \quad (\text{A1.10})$$

where μ_0 is vacuum magnetic permeability. λ' is an adimensional constant that makes H and M magnitudes of the same species. The source of magnetic field \vec{H} is magnetized matter, reason why it is more often used in this book than \vec{B} .

For the systems of units discussed here it happens that

$$\frac{4\pi k_2}{\mu_0 k_3^2 \lambda'} = 1. \quad (\text{A1.11})$$

The **magnetic dipole moment** of a plane coil of wire with electric current I that encircles an area A is

$$m = k_3 I \cdot A. \quad (\text{A1.12})$$

The constants appearing in **Maxwell's equations** are the following:

$$\begin{aligned}
\nabla \cdot \vec{D} &= \lambda \rho, & \nabla \times \vec{E} &= -k_3 \frac{\partial \vec{B}}{\partial t}, \\
\nabla \vec{B} &= 0, & \nabla \times \vec{H} &= k_3 \left(\lambda' \vec{J} + \frac{\partial \vec{D}}{\partial t} \right),
\end{aligned}
\tag{A1.13}$$

where ρ is the volume density of electric charge and \vec{J} the surface density of electric current.

For the SI system the given constants have the following values:

$$\begin{aligned}
k_1 &= \frac{1}{4\pi\epsilon_0} = 10^{-7} c^2 \frac{kg \cdot m^3}{C^2 s^2}, & k_2 &= \frac{\mu_0}{4\pi} = \frac{10^{-7} kg \cdot m}{C^2}, & k_3 &= 1, \\
\epsilon_0 &= \frac{10^7}{4\pi c^2} \frac{C^2 s^2}{kg \cdot m^3}, & \lambda &= 1, & \mu_0 &= 4\pi 10^{-7} \frac{kg \cdot m}{C^2}, & \lambda' &= 1.
\end{aligned}
\tag{A1.14}$$

The following table gives the values for all the common systems of electromagnetic units: **SI** or rationalized MKS, **ESU** or electrostatic CGS, **EMU** or electromagnetic CGS, **Gauss** and **HL** or Heaviside-Lorentz. Gauss system is the one preferred by physicist for the study of relativistic transformations because it makes explicit the speed of light c .

System	k_1	k_2	k_3	ϵ_0	λ	μ_0	λ'	$\frac{4\pi k_1 \epsilon_0}{\lambda}$	$\frac{4\pi k_2}{\mu_0 k_3 \lambda'}$
SI	$1/4\pi\epsilon_0$	$\mu_0/4\pi$	1	¹⁸⁴	1	$4\pi 10^{-7}$	1	1	1
ESU	1	$1/c^2$	1	1	4π	$1/c^2$	4π	1	1
EMU	c^2	1	1	$1/c^2$	4π	1	4π	1	1
Gauss	1	$1/c^2$	$1/c$	1	4π	1	4π	1	1
HL	$1/4\pi$	$1/4\pi c^2$	$1/c$	1	1	1	1	1	1

Table 4. The five common systems of electromagnetic units.

¹⁸⁴ See eq. A1.14.

Appendix 2: Vectorial Operators

The following properties of operator ∇ (*nabla* or *del*) are frequently used in this book¹⁸⁵.

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}. \quad (\text{A2.1})$$

$$\nabla \frac{1}{|\vec{r} - \vec{r}'|} = - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}. \quad (\text{A2.2})$$

$$\Delta = \nabla \cdot \nabla = \sum_{\alpha} \frac{\partial^2}{\partial x_{\alpha}^2}. \quad (\text{A2.3})$$

$$\begin{aligned} & \nabla [\vec{F}(\vec{r}) \cdot \vec{G}(\vec{r})] \\ &= (\vec{F}(\vec{r}) \cdot \nabla) \vec{G}(\vec{r}) + (\vec{G}(\vec{r}) \cdot \nabla) \vec{F}(\vec{r}) + \vec{F}(\vec{r}) \times (\nabla \times \vec{G}(\vec{r})) + \vec{G}(\vec{r}) \times (\nabla \times \vec{F}(\vec{r})). \end{aligned} \quad (\text{A2.4})$$

$$\nabla \cdot (f(\vec{r}) \vec{F}(\vec{r})) = f(\vec{r}) \nabla \cdot \vec{F}(\vec{r}) + \nabla f(\vec{r}) \cdot \vec{F}(\vec{r}). \quad (\text{A2.5})$$

$$\nabla \cdot (\vec{F}(\vec{r}) \times \vec{G}(\vec{r})) = \vec{G}(\vec{r}) \cdot \nabla \times \vec{F}(\vec{r}) - \vec{F}(\vec{r}) \cdot \nabla \times \vec{G}(\vec{r}). \quad (\text{A2.6})$$

$$\nabla \times \nabla f(\vec{r}) = 0. \quad (\text{A2.7})$$

$$\nabla \times (f(\vec{r}) \vec{F}(\vec{r})) = f(\vec{r}) (\nabla \times \vec{F}(\vec{r})) + (\nabla f(\vec{r})) \times \vec{F}(\vec{r}). \quad (\text{A2.8})$$

$$\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla) \vec{F} - (\vec{F} \cdot \nabla) \vec{G} + \vec{F} (\nabla \cdot \vec{G}) - \vec{G} (\nabla \cdot \vec{F}). \quad (\text{A2.9})$$

¹⁸⁵ Korn and Korn, pp. 157-162.

Appendix 3: Integral theorems of vector calculus

Laplacian of a point charge's potential

Eq. A3.7 that defines the trace of the internal depolarization tensor uses the value of the following expression:

$$\nabla \cdot \nabla \left(\iiint_V \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' \right) = \Delta I'(\vec{r}), \quad (\text{A3.1})$$

which coincides with $\Delta I(\vec{r})$ cuando $f(\vec{r}')=1$.

For $\vec{r} \neq \vec{r}'$ the order of integrating and taking derivatives may be permuted.

$$\begin{aligned} \Delta \iiint_V \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' &= \iiint_V f(\vec{r}') \Delta \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r', \\ \text{where } \Delta \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) &= \nabla \cdot \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -\nabla \cdot \left(\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) \\ &= -\frac{\nabla \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} - \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \right) \cdot (\vec{r} - \vec{r}') = -\frac{3}{|\vec{r} - \vec{r}'|^3} + 3 \frac{(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^5} = 0 \quad \forall \vec{r} \neq \vec{r}', \end{aligned} \quad (\text{A3.2})$$

where eq. A2.5 was used. It is thus proved that the laplacian vanishes for $\vec{r} \neq \vec{r}'$.

The case where a field point coincides with a source point —only possible inside the material body— is more difficult. It is not uncommon to find books on electromagnetism with erroneous proofs that by mere chance give the right value¹⁸⁶. That is why a proof is given here that circumvents the common pitfalls.

The order of calculation of the integral and the derivatives may be exchanged only when the integrand has integrable singularities¹⁸⁷. It is therefore convenient

¹⁸⁶ See, for instance Jackson, p. 13 and Reitz, p. 44, where they take the double derivative inside the integral. V. Hnizdo, Eur. J. Phys. **21**, pp. L1-L3 (2000) makes errors like using Gauss's Law as different from the divergence theorem.

¹⁸⁷ Hnizdo, ; *Generalized second-order partial derivatives of 1/r*: European Journal of Physics, vol. **32**, pp. 287-297; 2011.

to divide the body's volume in two parts: a spherical volume of radius R_0 around the singularity —radius that at the end will tend to 0— and the rest of the body, volume $V' = V - V_0$, as illustrated at [Figure 25](#).

That is, for \vec{r} fixed but arbitrary,

$$\begin{aligned} I'(\vec{r}) &= \iiint_V \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' \\ &= \iiint_{V-V_0} \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' + \iiint_{V_0} \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'. \end{aligned} \quad (\text{A3.3})$$

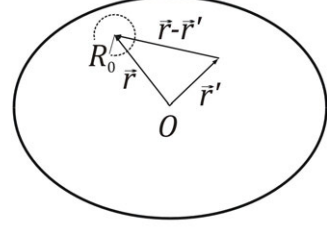


Figure 25. Isolating the singularity inside the body.

As shown before, the laplacian of the first integral vanishes so that the calculation is reduced to

$$\begin{aligned} \Delta \iiint_V \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' &= \Delta \iiint_{V_0} \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' \\ &= \nabla \cdot \nabla \iiint_{V_0} \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' = -\nabla \cdot \iiint_{V_0} f(\vec{r}') \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r'. \end{aligned} \quad (\text{A3.4})$$

When $I'(\vec{r})$ (eq. 3.8) is a continuous function with well behaved derivatives, which happens for any continuous and differentiable function $f(\vec{r}')$, the gradient theorem eq. A3.9 may be used to reduce the last volume integral to a surface one without singularities:

$$\begin{aligned} \nabla \cdot \iiint_{V_0} f(\vec{r}') \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r' &= \nabla \cdot \oint_{S_0} f(\vec{r}') \frac{d^2\vec{r}'}{|\vec{r} - \vec{r}'|} = \oint_{S_0} f(\vec{r}') \nabla \cdot \left(\frac{d^2\vec{r}'}{|\vec{r} - \vec{r}'|} \right) \\ &= \oint_{S_0} f(\vec{r}') \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cdot d^2\vec{r}' = - \oint_{S_0} f(\vec{r}') \frac{(\vec{r} - \vec{r}') \cdot d^2\vec{r}'}{R_0^3} = \oint_{S_0} f(\vec{r}') \frac{d^2r'}{R_0^2}. \end{aligned} \quad (\text{A3.5})$$

where $d^2\vec{r}'$ is the vectorial element of area.

The following limit is next taken:

$$\lim_{R_0 \rightarrow 0} \oint_{S_0} f(\vec{r}') \frac{d^2r'}{R_0^2} = f(\vec{r}) \lim_{R_0 \rightarrow 0} \oint_{S_0} \frac{d^2r'}{R_0^2} = 4\pi f(\vec{r}), \quad (\text{A3.6})$$

where use was made of eq. A2.5, of the fact that the vectors in the scalar product $(\vec{r} - \vec{r}') \cdot d^2\vec{r}'$ are antiparallel, and the definition of solid angle. At all steps the integrands have integrable singularities.

Therefore

$$\Delta \iiint_V \frac{f(\vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|} = \begin{cases} -4\pi f(\vec{r}) & \text{for } \vec{r} \in V \\ 0 & \text{for } \vec{r} \notin V \end{cases}, \quad (\text{A3.7})$$

Divergence, gradient and rotor theorems

The divergence or Gauss-Ostrogradsky's theorem establishes the following identity,^{188,189}

$$\iiint_V \nabla \cdot \vec{F}(\vec{r}) d^3 r = \oint_S \vec{F}(\vec{r}) \cdot d^2 \vec{r}, \quad (\text{A3.8})$$

where \hat{r} is the unit vector normal to surface S and outgoing from volume V . In its standard formulation the theorem requires that the field \vec{F} and its first derivatives have no singularities in the region V of integration and its boundary S . It is also valid for fields with integrable singularities. Such conditions are not fulfilled for the point charge field eq. 1.3.

The gradient theorem is a corollary of the divergence theorem^{190,191}

$$\iiint_V \nabla f(\vec{r}) d^3 r = \oint_S f(\vec{r}) d^2 \vec{r}. \quad (\text{A3.9})$$

Another corollary is the curl theorem^{192,193}:

$$\iiint_V \nabla \times \vec{F}(\vec{r}) d^3 r = - \oint_S \vec{F}(\vec{r}) \times d^2 \vec{r}. \quad (\text{A3.10})$$

Extension of the divergence theorem

The standard formulation of the divergence theorem is insufficient for applications to electromagnetism, where the field sources are often singularities like point charges, lines of charge, surface densities of charge and current, and step discontinuities of polarizations. The extension of the divergence theorem is done next for two types of singularities; it may also be done for the line of charge¹⁹⁴, but it is of no interest here.

¹⁸⁸ Korn and Korn, p. 163.

¹⁸⁹ Kuntzmann, pp. 329-332.

¹⁹⁰ Korn and Korn, p. 163.

¹⁹¹ Kuntzmann, p. 339.

¹⁹² Korn and Korn, p. 163.

¹⁹³ Kuntzmann, p. 340.

¹⁹⁴ Kuntzmann, p. 371.

Point charge type singularities

The extension of the divergence theorem for point charges¹⁹⁵, usually not claimed as different from the original one¹⁹⁶, is known in physics as the electrostatic Gauss's Law. It is applied to fields of the following kind:

$$\vec{F}(\vec{r}) = \sum_j \frac{q_j (\vec{r} - \vec{r}_j)}{|\vec{r} - \vec{r}_j|^3}. \quad (\text{A3.11})$$

In order to apply the standard divergence theorem it is necessary to exclude from the region V of integration an infinitesimal volume V_j around every singularity. The exclusions are indicated here by replacing volume V by V' , replacing the original integral by what mathematicians call *principal value integral*. The process is similar to that previously depicted at Figure 25, except that one must include the flux at the surfaces S_j that are the boundaries of each volume V_j . Therefore

$$\iiint_{V'} \nabla \cdot \vec{F}(\vec{r}) d^3r = \oiint_S \vec{F}(\vec{r}) \cdot d^2\vec{r} - \sum_j \oiint_{S_j} \frac{q_j (\vec{r} - \vec{r}_j) \cdot \hat{S}_j}{|\vec{r} - \vec{r}_j|^3} d^2r', \quad (\text{A3.12})$$

where the sum's negative sign comes from the fact that the vectors are entrant to the volume, not salient as required by the divergence theorem. Except for the factor q_j , all the integrals in the sum are equal because they subtend the same solid angle¹⁹⁷,

$$\oiint_{S_j} \frac{(\vec{r} - \vec{r}_j) \cdot d\vec{S}_j}{|\vec{r} - \vec{r}_j|^3} = \oiint_{S_j} \sin(\theta) d\theta d\phi = 4\pi, \quad (\text{A3.13})$$

where a spherical coordinate system is used.

Resulta finalmente

$$\iiint_{V'} \nabla \cdot \vec{F}(\vec{r}) d^3r = \oiint_S \vec{F}(\vec{r}) \cdot d^2\vec{r} - 4\pi \sum_j q_j. \quad (\text{A3.14})$$

In the formulation of the electrostatic Gauss's theorem, \vec{F} is the electric field \vec{E} and the equation is presented in the following way:

$$\oiint_S \vec{E}(\vec{r}) \cdot d^2\vec{r} = 4\pi k_1 \iiint_{V'} \rho(\vec{r}) d^3r + 4\pi k_1 \sum_j q_j, \quad (\text{A3.15})$$

where the relationship between \vec{E} and the volume charge density ρ has been used¹⁹⁸. The most general expression of the relationship between the flux of the

¹⁹⁵ Kuntzmann, p. 369.

¹⁹⁶ Reitz, p. 35.

¹⁹⁷ Kemmer, p. 71.

¹⁹⁸ Reitz, p. 85 eq. 4-29.

electric field and the distributions of free charge is obtained upon addition of the surface densities of charge (step discontinuities discussed in next section) and linear densities of charge (not done here). The polarization charges and currents are discussed in the main text, case by case

Step discontinuities

Generalized divergence theorem

The generalization is given here of the divergence theorem for the case in which the field has a step discontinuity through a surface¹⁹⁹, that in this book will be the body's surface.

A field is said to have a step discontinuity (also called *jump discontinuity* or *discontinuity of the first kind*) through a surface Σ when it is continuous and finite in any neighbourhood of Σ but its limit values over both sides of Σ are different. Defining conventionally a positive and a negative side of Σ ²⁰⁰ (see Figure 26) and identifying each limit with the corresponding upper + or - index, one has

$$\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r}) \neq 0. \quad (\text{A3.16})$$

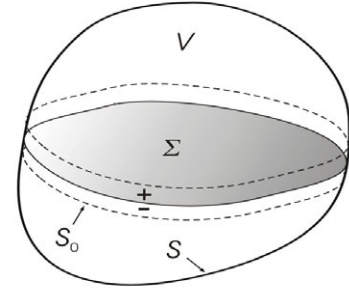


Figure 26. Step discontinuity surface Σ .

The divergence theorem eq. A3.8 cannot be applied to a region than contains a step discontinuity. The problem is not that the integrals involved cannot be computed for such surfaces²⁰¹, but the omission of the surface. Such integrals have to be divided and computed separately for each of the regions that have Σ as a boundary. In the case of Figure 26 the divergence theorem should not be applied for the whole of volume V , but to the two regions determined by Σ . A mathematically rigorous way of justifying this is to exclude an infinitesimal volume V_0 around Σ , volume delimited by the surface S_0 identified in the figure with dashed lines.

The application of the theorem in the indicated way gives

$$\iiint_{V-V_0} \nabla \cdot \vec{F}(\vec{r}) d^3r = \oint_S \vec{F}(\vec{r}) \cdot d\vec{S} + \oint_{S_0} \vec{F}(\vec{r}) \cdot d\vec{S}. \quad (\text{A3.17})$$

When the vector field vectorial is finite in any neighbourhood of Σ , so it is the flux on the edge of surface S_0 también lo is and tends to 0 when the thickness of V_0 goes to 0. That is, the principal value integral should be computed, which will be indicated in this book with the specification $\lim V_0 \rightarrow 0$. Taking this limit does not modify the volume integral's value.

¹⁹⁹ Kuntzmann, pp. 371-372.

²⁰⁰ En este libro Σ is la superficie del cuerpo and el valor positivo del campo is el exterior a ella.

²⁰¹ See, for instance, Morse and Feshbach, p. 34.

In the aforesaid limit the previous equation gives

$$\begin{aligned}
 \iiint_V \nabla \cdot \vec{F}(\vec{r}) d^3r &= \lim_{V_0 \rightarrow 0} \iiint_{V-V_0} \nabla \cdot \vec{F}(\vec{r}) d^3r \\
 &= \oint_S \vec{F}(\vec{r}) \cdot d\vec{S} - \iint_{\Sigma} \vec{F}^+(\vec{r}) \cdot d\vec{\Sigma} + \iint_{\Sigma} \vec{F}^-(\vec{r}) \cdot d\vec{\Sigma} \\
 &= \oint_S \vec{F}(\vec{r}) \cdot d\vec{S} - \iint_{\Sigma} (\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r})) \cdot d\vec{\Sigma}
 \end{aligned}$$

Therefore,

$$\iiint_V \nabla \cdot \vec{F}(\vec{r}) d^3r = \oint_S \vec{F}(\vec{r}) \cdot d\vec{S} - \oint_{\Sigma} (\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r})) \cdot d\vec{\Sigma}. \quad (\text{A3.18})$$

where $d\vec{\Sigma}$ is a vector outgoing from the positive side of the surface. The equation is the generalization of the divergence theorem for the case where the field has a step discontinuity over a closed or open surface Σ .

A more compact and ‘‘elegant’’ demonstration may be given by using distribution theory, but this requires many other concepts, as an extension of the derivative operation²⁰² which exceeds by far the requisite of a first course of vector analysis imposed in this book.

Generalized rotor theorem

If in theorem A3.18 \vec{F} is replaced by $\vec{F} \times \vec{c}$, where \vec{c} is an arbitrary constant vector, one gets

$$\begin{aligned}
 \iiint_V \nabla \cdot (\vec{F}(\vec{r}) \times \vec{c}) d^3r &= \vec{c} \cdot \iiint_V \nabla \times \vec{F}(\vec{r}) d^3r \\
 &= \oint_S (\vec{F}(\vec{r}) \times \vec{c}) \cdot d\vec{S} - \oint_{\Sigma} (\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r})) \times \vec{c} \cdot d\vec{\Sigma} \\
 &= \vec{c} \cdot \oint_S (d\vec{S} \times \vec{F}(\vec{r})) - \vec{c} \cdot \oint_{\Sigma} d\vec{\Sigma} \times (\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r})),
 \end{aligned} \quad (\text{A3.19})$$

where eq. A2.6 and the cyclic property of the scalar triple product were used. As \vec{c} is an arbitrary vector, it is obtained

$$\iiint_V \nabla \times \vec{F}(\vec{r}) d^3r = - \oint_S \vec{F}(\vec{r}) \times d\vec{S} + \oint_{\Sigma} (\vec{F}^+(\vec{r}) - \vec{F}^-(\vec{r})) \times d\vec{\Sigma}, \quad (\text{A3.20})$$

Polarized bodies

All the bodies studied in this book are characterized by a polarization vector \vec{Q} with the following properties:

- It is uniform and non-vanishing inside the body V ;
- it vanishes outside the body;

²⁰² See, for instance, Farassat, F., *Introduction to Generalized Functions with applications in Aerodynamics and Aeroacoustics*, NASA, Langley Research Center, April 1996.

- it has a variable step discontinuity through the body's closed surface S , the S surface of the previous theorems;

If the integration region is V' with frontier S' that contains both V and Σ , it is obtained for eq. [A3.18](#)

$$\begin{aligned} \iiint_{V'} \nabla \cdot \vec{F}(\vec{r}) d^3r &= \iiint_V \nabla \cdot \vec{F}(\vec{r}) d^3r \\ &= \oint_{S'} \vec{F}(\vec{r}) \cdot d\vec{S} + \oint_{\Sigma} \vec{F}^-(\vec{r}) \cdot d\vec{\Sigma} = \oint_{\Sigma} \vec{F}^-(\vec{r}) \cdot d\vec{\Sigma}, \end{aligned} \quad (\text{A3.21})$$

because $\nabla \cdot \vec{F}$ and \vec{F} vanish outside V .

In the same way, eq. [A3.20](#) reduces to

$$\iiint_V \nabla \times \vec{F}(\vec{r}) d^3r = - \oint_{\Sigma} \vec{F}^-(\vec{r}) \times d\vec{\Sigma}. \quad (\text{A3.22})$$

Appendix 4: Field of an electric dipole

General expression

The electric field $\vec{E}(\vec{r})$ generated at point field \vec{r} by a body with electric dipole moment \vec{p} placed at the origin is

$$\boxed{\vec{E}(\vec{r}) = k_1 \frac{3(\vec{p} \cdot \vec{r})\vec{r} - r^2 \vec{p}}{r^3}} \quad (\text{A4.1})$$

The field \vec{E}^j generated by a dipole \vec{p}^j located at source point \vec{r}^j is²⁰³

$$\begin{aligned} \vec{E}^j(\vec{r}) &= k_1 \frac{3\vec{d}^j(\vec{r})(\vec{d}^j(\vec{r}) \cdot \vec{p}^j) - d^j(r)^2 \vec{p}^j}{d^j(r)^5}, \\ &= k_1 \frac{3\vec{d}^j(\vec{r})\vec{d}^j(\vec{r}) - d^j(r)^2 \mathbf{1}}{d^j(r)^5} \cdot \vec{p}^j, \quad \text{where } d^j(\vec{r}) = \vec{r} - \vec{r}^j, \end{aligned} \quad (\text{A4.2})$$

and $\mathbf{1}$ is the unit dyadic such that for an arbitrary vector \vec{C} it is

$$\mathbf{1} \cdot \vec{C} = \vec{C}. \quad (\text{A4.3})$$

The upper index indicates the source point and the argument the field point (see p. 11).

In matrix representation the previous expression becomes

$$\mathbf{E}(\vec{r}) = k_1 \mathbf{m}^j \cdot \mathbf{p}^j, \quad (\text{A4.4})$$

²⁰³ Reitz, p. 39 eq. 2-36.

where

$$\mathbf{m}^j(\vec{r}) = \begin{pmatrix} \frac{3d_x^j(\vec{r})^2 - d^j(\vec{r})^2}{d^j(\vec{r})^5} & \frac{3d_x^j(\vec{r})d_y^j(\vec{r})}{d^j(\vec{r})^5} & \frac{3d_x^j(\vec{r})d_z^j(\vec{r})}{d^j(\vec{r})^5} \\ \frac{3d_y^j(\vec{r})d_x^j(\vec{r})}{d^j(\vec{r})^5} & \frac{3d_y^j(\vec{r})^2 - d^j(\vec{r})^2}{d^j(\vec{r})^5} & \frac{3d_y^j(\vec{r})d_z^j(\vec{r})}{d^j(\vec{r})^5} \\ \frac{3d_z^j(\vec{r})d_x^j(\vec{r})}{d^j(\vec{r})^5} & \frac{3d_z^j(\vec{r})d_y^j(\vec{r})}{d^j(\vec{r})^5} & \frac{3d_z^j(\vec{r})^2 - d^j(\vec{r})^2}{d^j(\vec{r})^5} \end{pmatrix}. \quad (\text{A4.5})$$

Dipolar field's energy

A polarizable sphere of radius R immersed in an applied uniform field \vec{C}^0 generates outside a dipolar field $\vec{C}^q(\vec{r})$. Depending on the kind of material (dielectric, conductor, diamagnetic, paramagnetic or superconductor) and applied field (electric or magnetic) the constants and the dipolar moments will vary. Nevertheless, in all cases, when the center of the sphere is the origin of the coordinate system, the energy stored in the external field is proportional to the following integral:

$$\iiint_{V'} \vec{C}^q(\vec{r}) \cdot \vec{C}^q(\vec{r}) d^3r, \quad (\text{A4.6})$$

where V' is the complementary region of V (the one occupied by the spherical body) and

$$\vec{C}^q(\vec{r}) = \frac{3(\vec{q} \cdot \vec{r})\vec{r} - r^2\vec{q}}{r^5}. \quad (\text{A4.7})$$

The integrand is then

$$\begin{aligned} \vec{C}^q(\vec{r}) \cdot \vec{C}^q(\vec{r}) &= \frac{(3(\vec{q} \cdot \vec{r})\vec{r} - r^2\vec{q}) \cdot (3(\vec{q} \cdot \vec{r})\vec{r} - r^2\vec{q})}{r^{10}} \\ &= \frac{9r^2(\vec{q} \cdot \vec{r})^2 - 6r^2(\vec{q} \cdot \vec{r})^2 + q^2r^4}{r^{10}} = \frac{3(\vec{q} \cdot \hat{r})^2 + q^2}{r^6}. \end{aligned} \quad (\text{A4.8})$$

In an spherical coordinate system with z axis parallel to \vec{q}

$$\begin{aligned} \iiint_{V'} \vec{C}^q(\vec{r}) \cdot \vec{C}^q(\vec{r}) d^3r &= \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_R^\infty r^2 C^q(\vec{r})^2 dr \\ &= 2\pi \int_0^\pi \sin\theta d\theta \int_R^\infty r^{-4} (3(\vec{q} \cdot \hat{r})^2 + q^2) dr \\ &= 2\pi q^2 \left(\int_0^\pi (3\cos^2\theta + 1) \sin\theta d\theta \right) \left(\int_R^\infty r^{-4} dr \right). \end{aligned} \quad (\text{A4.9})$$

As

$$\begin{aligned} \int_0^\pi (3\cos^2\theta + 1) \sin\theta d\theta &= \int_\pi^0 (3\cos^2\theta + 1) d(\cos\theta) \\ &= \left[\cos^3\theta + \cos\theta \right]_\pi^0 = 4, \quad \int_R^\infty r^{-4} dr = \frac{r^{-3}}{3} \Big|_R^\infty = \frac{1}{3R^3}, \end{aligned} \quad (\text{A4.10})$$

therefore

$$\iiint_{V'} \vec{C}^q(\vec{r}) \cdot \vec{C}^q(\vec{r}) d^3r = \frac{2}{3} \frac{4\pi}{R^3} q^2 = 2 \left(\frac{4\pi}{3} \right)^2 \frac{q^2}{V}, \quad (\text{A4.11})$$

where V is the volume of the sphere and q the dipole moment norm.

Appendix 5:

Simmetries of the susceptibility tensors of single crystals

Electric and magnetic susceptibilities, as any tensorial property, have simmetries that reflect those of the single crystal they characterize. As illustrated by the depolarization tensor eq. 3.23, the relationship between components are a consequence of the tensor's transformation properties. The following table gives the relationships for all crystalline systems²⁰⁴ of the components of a single crystal's susceptibility tensor²⁰⁵, where the number of independent ones are given between parentheses in the first column.

cristalline system	clasification	principal coordinates	matrix representation
Cubic (1)	isotropic or anaxial	any one	$\begin{pmatrix} \chi & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{pmatrix}$
trigonal tetragonal hexagonal (2)	uniaxial	symmetry of revolution around axis of maximum symmetry	$\begin{pmatrix} \chi_{\perp} & 0 & 0 \\ 0 & \chi_{\perp} & 0 \\ 0 & 0 & \chi_{\parallel} \end{pmatrix}$
Orthorhombic (3)	biaxial	coordinate axes parallel to the three binary axis	$\begin{pmatrix} \chi_{\perp} & 0 & 0 \\ 0 & \chi_{\perp} & 0 \\ 0 & 0 & \chi_{\parallel} \end{pmatrix}$
Monoclinic (4)	biaxial	1 coordinate axis parallel to the single binary axis	$\begin{pmatrix} \chi_{11} & 0 & \chi_{13} \\ 0 & \chi_{22} & 0 \\ \chi_{13} & 0 & \chi_{33} \end{pmatrix}$
Triclinic (6)	biaxial	none	$\begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{31} \\ \chi_{12} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{23} & \chi_{33} \end{pmatrix}$

Table 5. Simmetries of the susceptibility tensor.

²⁰⁴ Dekker, Capítulo 1.

²⁰⁵ Nye, p. 23 Table 3.

Appendix 6: Dyadics

A dyadic is a mathematical entity —a linear combination of external products of unit vectors— such that its scalar product with a vector is a vector. Dyadics may be interpreted as vector operators such that its action (scalar product) on a vector gives another vector. In this book dyadics are written in boldface italics. The expression of \mathbf{n} , the dyadic most often quoted here, is

$$\mathbf{n}(\vec{r}) = \sum_{j,k} \hat{x}_j n_{jk}(\vec{r}) \hat{x}_k. \quad (\text{A6.1})$$

A particular case of dyadic is the external product of two vectors:

$$\mathbf{c} = \vec{A}\vec{B} = \sum_{j,k} A_j \hat{x}_j B_k \hat{x}_k. \quad (\text{A6.2})$$

Such is the case of the dyadic $\hat{\mathbf{s}}\hat{\mathbf{s}}$ in eq. 3.33.

The scalar product of a vector with a dyadic may be taken in two different ways:

$$\begin{aligned} \mathbf{n} \cdot \vec{A} &= \sum_{j,k} \hat{x}_j n_{jk} \hat{x}_k \cdot \sum_l \hat{x}_l A_l = \sum_{j,k,l} \hat{x}_j n_{jk} \hat{x}_k \cdot \hat{x}_l A_l \\ &= \sum_{j,k,l} \hat{x}_j n_{jk} \delta_{kl} A_l = \sum_{j,k} \hat{x}_j n_{jk} A_k = \sum_j \hat{x}_j \sum_k n_{jk} A_k; \\ \vec{A} \cdot \mathbf{n} &= \sum_j A_j \hat{x}_j \cdot \sum_{k,l} \hat{x}_k \hat{x}_l n_{kl} = \sum_{j,k,l} A_j \hat{x}_j \cdot \hat{x}_k \hat{x}_l n_{kl} \\ &= \sum_{j,k,l} A_j \delta_{jk} n_{kl} \hat{x}_l = \sum_{j,l} A_j n_{jl} \hat{x}_l = \sum_{j,k} A_k n_{kj} \hat{x}_j = \sum_j \hat{x}_j \sum_k n_{kj} A_k. \end{aligned} \quad (\text{A6.3})$$

The symmetric unit dyadic, here written $\mathbf{1}$, is

$$\mathbf{1} = \sum_j \hat{x}_j \hat{x}_j = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}, \quad (\text{A6.4})$$

such that

$$\mathbf{1} \cdot \vec{A} = \sum_j \hat{x}_j \hat{x}_j \cdot \sum_k \hat{x}_k A_k = \sum_{j,k} \hat{x}_j \hat{x}_j \cdot \hat{x}_k A_k = \sum_{j,k} \hat{x}_j \delta_{jk} A_k = \sum_j \hat{x}_j A_j = \vec{A}; \quad (\text{A6.5})$$

$$\vec{A} \cdot \mathbf{1} = \vec{A}.$$

In general

$$\mathbf{n} \cdot \vec{A} \neq \vec{A} \cdot \mathbf{n} \quad (\text{A6.6})$$

In the same fashion, exchanging the order of vectors modifies the value of external products. For instance,

$$\mathbf{a} = \sum_{j,k} \hat{x}_j \hat{x}_k a_{jk} \neq \sum_{j,k} \hat{x}_k \hat{x}_j a_{jk} = \sum_{j,k} \hat{x}_j \hat{x}_k a_{kj} = \mathbf{a}^t, \quad (\text{A6.7})$$

where the last dyadic is the transposed of the first.

A dyadic \mathbf{a} is symmetric when it is equal to its transposed dyadic, that is, when its elements a_{jk} are symmetric:

$$\mathbf{a} = \sum_{j,k} \hat{x}_j \hat{x}_k a_{jk} = \mathbf{a}^t = \sum_{j,k} \hat{x}_j \hat{x}_k a_{kj} \quad \text{si} \quad a_{jk} = a_{kj}. \quad (\text{A6.8})$$

The two ways of taking the scalar product coincide only for symmetric dyadics:

$$\mathbf{n} \cdot \bar{\mathbf{A}} = \sum_j \hat{x}_j \sum_k n_{jk} A_k = \sum_j \hat{x}_j \sum_k n_{kj} A_k = \bar{\mathbf{A}} \cdot \mathbf{n}. \quad (\text{A6.9})$$

When a dyadic is represented by a square matrix —here written in roman boldface— the two ways of taking the scalar product correspond to different representations of the vectors:

$$\begin{aligned} \text{If } \mathbf{a} &= \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \\ \mathbf{A}^t \cdot \mathbf{a} &= \begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \cdot \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix} = \begin{pmatrix} A_x a_{xx} + A_y a_{yx} + A_z a_{zx} \\ A_x a_{xy} + A_y a_{yy} + A_z a_{zy} \\ A_x a_{xz} + A_y a_{yz} + A_z a_{zz} \end{pmatrix}, \quad (\text{A6.10}) \\ \mathbf{a} \cdot \mathbf{A} &= \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} a_{xx} A_x + a_{xy} A_y + a_{xz} A_z \\ a_{yx} A_x + a_{yy} A_y + a_{yz} A_z \\ a_{zx} A_x + a_{zy} A_y + a_{zz} A_z \end{pmatrix}. \end{aligned}$$

The trace of a dyadic, of special interest for the depolarization tensor, is defined as the sum of its diagonal elements. That is

$$\text{Tr } \mathbf{a} = \sum_j a_{jj} = a_{xx} + a_{yy} + a_{zz}. \quad (\text{A6.11})$$

For dyadics defined as the external product of two vectors, the trace coincides with the scalar product:

$$\text{If } \mathbf{a} = \bar{\mathbf{A}} \bar{\mathbf{B}}, \quad \text{Tr } \mathbf{a} = \sum_j a_{jj} = \sum_j A_j B_j = A_x B_x + A_y B_y + A_z B_z = \bar{\mathbf{A}} \cdot \bar{\mathbf{B}}. \quad (\text{A6.12})$$

Appendix 7: Ellipsoids

Equations and main features

Equations, sections, volume and surface

In an arbitrary orthogonal system of coordinates the matrix equation of an ellipsoidal surface²⁰⁶ is²⁰⁷

$$\mathbf{r}^t \cdot \mathbf{A}^{-1} \cdot \mathbf{r} = 1, \text{ where } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix}, \quad (\text{A7.1})$$

where \mathbf{A} is a positive definite symmetric matrix with eigenvalues $1/a^2$, $1/b^2$ and $1/c^2$. The coordinate system that diagonalizes \mathbf{A} , here called the principal system, is the one with origin at the center of the ellipsoid that coincides with its principal axes²⁰⁸. Equation A7.1 then becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (\text{A7.2})$$

where a , b and c are the lengths of the ellipsoid semiaxes.

The parametric equations of the ellipsoid are²⁰⁹

$$\begin{aligned} x &= a \cdot \cos u \cdot \sin v, & y &= b \cdot \sin u \cdot \sin v, & z &= c \cdot \cos v, \\ \text{where } 0 \leq u \leq 2\pi, & 0 \leq v \leq \pi, \end{aligned} \quad (\text{A7.3})$$

where the definition of the angles u , v and their domains are identical to the angles φ and θ of a spherical coordinate system, but not its relationships with the components of the position vector \vec{r} .

The intersection of the ellipsoid with a plane is always an ellipse²¹⁰ that becomes two overlapping circumferences for spheroids.

²⁰⁶ Those not familiar with actual tri-dimensional ellipsoids can profit from the interactive views given by the Wolfram Demonstration Project [Ellipsoids](#) by Jeff Bryant.

²⁰⁷ Korn and Korn, pp. 74-82.

²⁰⁸ Korn and Korn, p. 43.

²⁰⁹ [Ellipsoid](#) in Wolfram MathWorld, eqs. 3-5.

The ellipsoid's volume V is

$$V = \frac{4\pi}{3}abc. \quad (\text{A7.4})$$

The surface area S is

$$S = 2\pi c^2 + 2\pi \frac{ab}{\sin \phi} \left(E(\phi; k) \sin^2 \phi + F(\phi; k) \cos^2 \phi \right), \quad (\text{A7.5})$$

$$\text{donde } a \geq b \geq c, \quad \cos \phi = \frac{c}{a}, \quad k^2 = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)},$$

where $F(\phi, k)$ and $E(\phi, k)$ are the Legendre's incomplete elliptic integrals of the first and second kind discussed in [Appendix 9](#).

Aspect ratios

In daily life, a sphere is called a sphere regardless of the radius' value, because its apparent size varies with the distance to the observer. In mathematical language it is said that all spheres are similar, the single parameter that characterize them differs by an arbitrary multiplication factor k . When two parameters characterize the shape, as in a rectangle of sides a and b , an observer sees two similar ones (sides $k \cdot a$ and $k \cdot b$) as having the same aspect. If $a \leq b$, the ratio $\beta = b/a$ is called the aspect ratio, where the equal sign characterizes the square as a particular case. By definition an aspect ratio has the following range of values,

$$0 \leq \beta \leq 1. \quad (\text{A7.6})$$

It may seem that the value 0 should be excluded, but it is convenient to include it in order to characterize the degenerate rectangle with $a = \infty$, two parallel lines.

In the case of ellipsoids, three parameters a, b, c characterize its shapes. If $a \geq b \geq c$, there are two aspect ratios,

$$0 \leq \beta = b/a \leq 1, \quad 0 \leq \gamma = c/a \leq 1. \quad (\text{A7.7})$$

The different shapes of ellipsoids according to its aspect ratios are identified in [Table 3](#). Notice that the aspect ratios of the ellipsoid do not suffice to tell apart a right circular cylinder of infinite length from an elliptic one. The electrostatic and magnetostatic properties of ellipsoidal bodies of the same aspect is exactly the same because similar ellipsoids have the same depolarization tensor (see section [N\(0\) is determined by aspect ratios](#)).

²¹⁰ G. F. Childe, *Singular Properties of the Ellipsoid and Associated Surfaces of the Nth Degree*, Macmillan, 1861, pp. 133-137.

Normals to the surface and central distances

The unit vector $\hat{s}(\vec{r})$ normal to surface S at point \vec{r} may be obtained from the gradient²¹¹ of the eq. A7.2:

$$\hat{s}(\vec{r}) = \frac{\nabla \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)}{\left| \nabla \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \right|} = \frac{\frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \quad (\text{A7.8})$$

It may, for instance, be easily checked that this vector is normal to the surface of the ellipsoid at the intersections with each of the three cartesian axes and on the ellipses determined by the planes defined by any two cartesian axes²¹² (see Problem 29).

Unit vector \hat{s} has some useful properties, best derived when it is written in terms of the following normal vector $\vec{s}(\vec{r})$:

$$\hat{s}(\vec{r}) = \frac{\vec{s}(\vec{r})}{|\vec{s}(\vec{r})|}, \quad \text{where} \quad \vec{s}(\vec{r}) = \frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z}. \quad (\text{A7.9})$$

Then, if \vec{r} is the position vector of a point on the ellipsoidal surface, equation A7.2 is equivalent to the vector equation

$$\begin{aligned} \vec{r}_1 \cdot \vec{s}(\vec{r}_1) &= \left(x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z} \right) \cdot \left(\frac{x_1}{a^2} \hat{x} + \frac{y_1}{b^2} \hat{y} + \frac{z_1}{c^2} \hat{z} \right) \\ &= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad \forall \vec{r}_1 \in S. \end{aligned} \quad (\text{A7.10})$$

The equation of the plane tangent to the ellipsoidal surface at point \vec{r}_1 is²¹³

$$\vec{r} \cdot \vec{s}(\vec{r}_1) = \frac{x_1}{a^2} x + \frac{y_1}{b^2} y + \frac{z_1}{c^2} z = 1, \quad (\text{A7.11})$$

relationship that may be easily checked to be true at the intersections of the three principal axes with the ellipsoidal surface.

In a similar fashion, vectors \vec{r}_1 and $\vec{s}(\vec{r}_1)$ determine a family of parallel planes whose equation is

$$\vec{r} \cdot \vec{r}_1 \times \hat{s}(\vec{r}_1) = k, \quad (\text{A7.12})$$

²¹¹ Korn and Korn, p. 158 eq. 5.5-5.

²¹² Korn and Korn, p. 46.

²¹³ Kemmer, p. 11 Problem 16 and p. 196.

where k is constant for each plane and equal to its distance to the origin. The plane through point \vec{r}_1 satisfies the equation

$$\vec{r}_1 \cdot \vec{r}_1 \times \vec{s}(\vec{r}_1) = \vec{r}_1 \times \vec{r}_1 \cdot \vec{s}(\vec{r}_1) = 0, \quad (\text{A7.13})$$

expression that shows that \vec{r}_1 , $\vec{s}(\vec{r}_1)$ and the center O of the ellipsoid lie on the same plane.

In what follows a *central plane* of the ellipsoid will be any plane that contains its center, the origin its principal system of coordinates. In the same fashion, a *central section* of the ellipsoid will be any intersection of the surface with a central plane, as illustrated in Figure 27.

The geometrical meaning of vector $\vec{s}(\vec{r})$ may be obtained from Figure 27. It is there seen that the distance d_j from the ellipsoid's center O to the plane tangent to S at point \vec{r}_j is the projection of that vector on the direction of \hat{s}_j . This value is here called the *central distance* corresponding to the given point of the ellipsoidal surface. The dot lines in the figure correspond to the intersections of the tangent planes and the plane that determines the depicted central section.

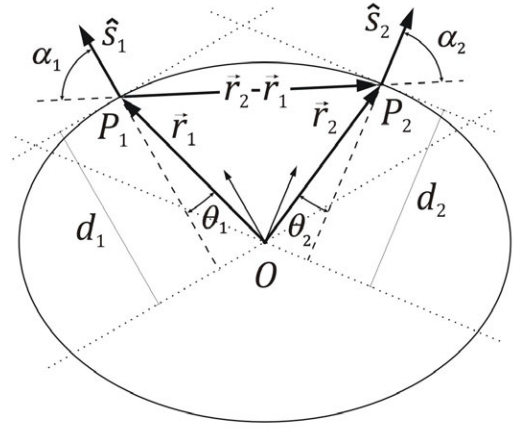


Figure 27. Central distance and chord in an ellipsoid's central section.

The general expression for the central distance d of a point \vec{r} on the ellipsoidal surface is

$$d(\vec{r}) = r \cos \theta = \vec{r} \cdot \hat{s}(\vec{r}) = \frac{\vec{r} \cdot \vec{s}(\vec{r})}{s(\vec{r})} \quad \text{for } \vec{r} \in S. \quad (\text{A7.14})$$

Using eqs. A7.8 and A7.10, the central distance turns out to be

$$\begin{aligned} d(\vec{r}) &= \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-1/2} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-1/2} = \frac{1}{s(\vec{r})} \\ &= \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}} = \frac{a^2 b^2 c^2}{\sqrt{b^4 c^4 x_1^2 + a^4 c^4 y_1^2 + a^4 b^4 z_1^2}}, \end{aligned} \quad (\text{A7.15})$$

Equation A7.15 shows that the norm of vector $\vec{s}(\vec{r})$ is the inverse of the central distance of point \vec{r} on the ellipsoidal surface. A simple verification of the formula is to apply it to the sphere, where $a = b = c = R = d$.

Eq. A7.14 determines the angle between vector \vec{r}_1 of a point P_1 on the ellipsoidal surface and the normal to it at that point:

$$\cos\theta_1 = \frac{\vec{r}_1 \cdot \hat{s}(\vec{r}_1)}{r_1}. \quad (\text{A7.16})$$

The following dyadic is one of the components of the external depolarization tensor eq. 4.66:

$$\hat{s}(\vec{r}|\kappa)\hat{s}(\vec{r}|\kappa) = \left(\frac{x^2}{(a^2+\kappa)^2} + \frac{y^2}{(b^2+\kappa)^2} + \frac{z^2}{(c^2+\kappa)^2} \right)^{-1} \sum_{\alpha,\beta} \frac{\hat{x}_\alpha x_\alpha x_\beta \hat{x}_\beta}{(d_\alpha^2+\kappa)(d_\beta^2+\kappa)}, \quad (\text{A7.17})$$

where $\alpha = x, y, z$, $d_x = a, d_y = b, d_z = c$.

This tensor has the property that when applied to a vector leaves only the component normal to the ellipsoidal surface eq. A7.20 (see also section [Surface step discontinuity](#), in particular eq. 3.39).

The following property is responsible of the vanishing trace of the external depolarization tensor eq. :

$$\begin{aligned} \text{Tr } \hat{s}(\vec{r}|\kappa)\hat{s}(\vec{r}|\kappa) &= \left(\frac{x^2}{(a^2+\kappa)^2} + \frac{y^2}{(b^2+\kappa)^2} + \frac{z^2}{(c^2+\kappa)^2} \right)^{-1} \sum_{\alpha} \frac{x_\alpha^2}{(d_\alpha^2+\kappa)^2} \\ &= \left(\frac{x^2}{(a^2+\kappa)^2} + \frac{y^2}{(b^2+\kappa)^2} + \frac{z^2}{(c^2+\kappa)^2} \right)^{-1} \left(\frac{x^2}{(a^2+\kappa)^2} + \frac{y^2}{(b^2+\kappa)^2} + \frac{z^2}{(c^2+\kappa)^2} \right) = 1. \end{aligned} \quad (\text{A7.18})$$

Summary of properties

The given properties of normal vector $\vec{s}(\vec{r})$ are collected next.

$$\begin{aligned} \vec{s}(\vec{r}) &= \frac{x}{a^2} \hat{x} + \frac{y}{b^2} \hat{y} + \frac{z}{c^2} \hat{z}: \quad \vec{s}(\vec{r}) \perp S \text{ at } \vec{r}; \\ \vec{r} \cdot \vec{s}(\vec{r}_1) &= \frac{x_1}{a^2} x + \frac{y_1}{b^2} y + \frac{z_1}{c^2} z = 1 \quad \text{is the plane tangent to } S \text{ at } \vec{r}_1; \\ \vec{r}_1 \cdot \vec{s}(\vec{r}_1) &= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \quad \forall \vec{r}_1 \in S; \\ \frac{1}{s(\vec{r})} &= d(\vec{r}) = \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}} = \frac{a^2 b^2 c^2}{\sqrt{b^4 c^4 x_1^2 + a^4 c^4 y_1^2 + a^4 b^4 z_1^2}} \end{aligned} \quad (\text{A7.19})$$

is the distance to the ellipsoid's center of the plane tangent at \vec{r} ;
 $\hat{s}\hat{s}$ is the dyadic that projects from any vector the component normal to the surface S of the ellipsoid.

Confocal ellipsoids

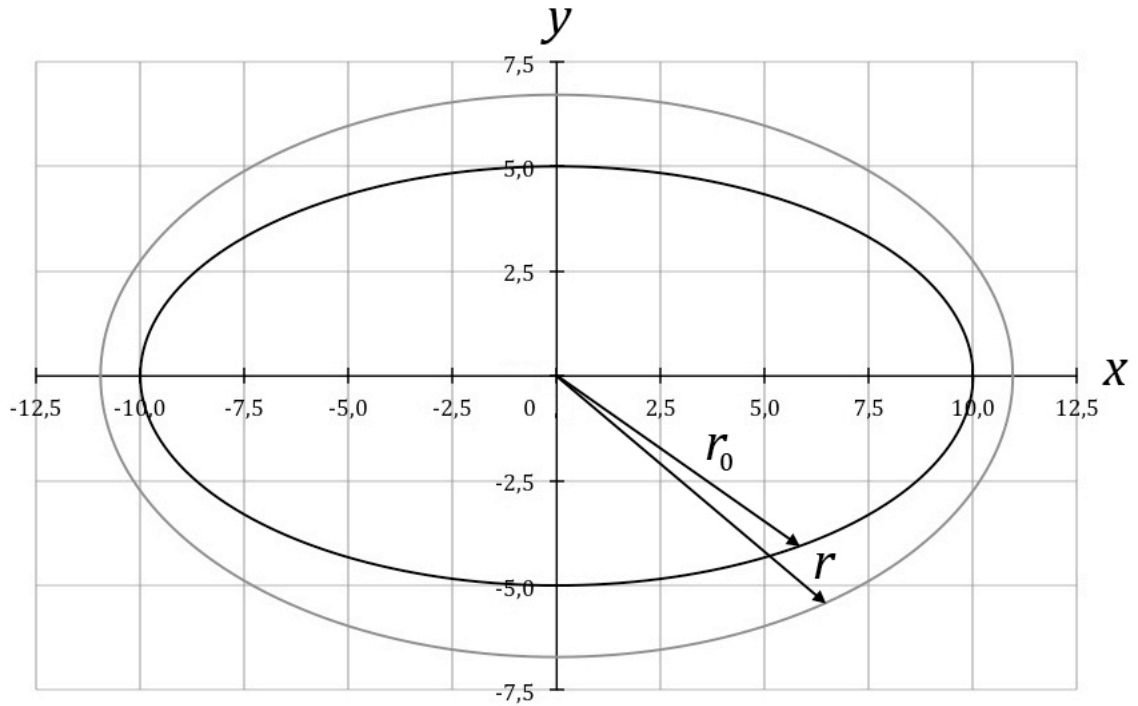


Figure 28. Central section of two confocal ellipsoids.

For the calculation of the potential outside the body use is made of a family of ellipsoids confocal with the body's surface,

$$\frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1, \quad \text{where } \kappa \geq 0, \quad \vec{r} \notin V. \quad (\text{A7.20})$$

Figure 28 shows the central section $z = 0$ of two such ellipsoids, where $a = 5$, $b = 10$ and $\kappa = 20$. Notice that the confocal ellipse is always rounder than the original one and that the distance between surfaces is largest for the smallest semiaxes. κ is usually determined as the largest or positive algebraic root of eq. A7.20, where \vec{r} is any field point outside the surface of the ellipsoidal body eq. A7.23. To the author's knowledge no explicit expressions have been given for these roots, which in general are calculated point by point²¹⁴ by methods like the Newton-Raphson iterations²¹⁵.

Physical-geometric interpretation of κ

The unit of the confocal parameter κ is *length*², which suggests its interpretation as a squared distance. This assumption is confirmed when eq. A7.20 is solved for the only type of spheroid where an explicit solution may be easily found, a sphere of radius R (see eq. 6.81). In this case

²¹⁴ See, for instance, at Barczak & Breit & Jusiel, *Ellipsoids, material points and material segments*, comment in the paragraph preceding eq. 70 in an un-numbered page.

²¹⁵ Korn & Korn, section 20.2-1.

$$\kappa = r^2 - R^2. \quad (\text{A7.21})$$

For the general ellipsoid eq. A7.20, upper and lower bounds for κ may be found. The squared distance from its center to any surface's point satisfies the following inequality,

$$c^2 + \kappa \leq r^2 \leq a^2 + \kappa, \text{ so that } r^2 - a^2 \leq \kappa \leq r^2 - c^2. \quad (\text{A7.22})$$

If \vec{r}_0 is a point on the body's surface, its coordinates satisfy the equation

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1, \quad (\text{A7.23})$$

its confocal ellipsoid being given by eq. A7.20. The maximum absolute value of each variable is its semiaxis, when the two other variables are zero. For x , for instance, it is then obtained

$$\kappa = x^2 - x_0^2 = (a^2 + \kappa) - a^2, \quad (\text{A7.24})$$

and a similar result is obtained for y and z suggesting that the value of κ may be $r^2 - r_0^2$. The conjecture is true when the components of \vec{r} and \vec{r}_0 are taken to have equal angles in their parametric expressions eqs. A7.3:

$$\begin{aligned} x_0^2 &= a^2 \cdot \cos^2 u_0 \cdot \sin^2 v_0, & x^2 &= (a^2 + \kappa) \cdot \cos^2 u_0 \cdot \sin^2 v_0 \\ y_0^2 &= b^2 \cdot \sin^2 u_0 \cdot \sin^2 v_0, & y^2 &= (b^2 + \kappa) \cdot \sin^2 u_0 \cdot \sin^2 v_0, \\ z_0^2 &= c^2 \cdot \cos^2 v_0, & z^2 &= (c^2 + \kappa) \cdot \cos^2 v_0. \end{aligned} \quad (\text{A7.25})$$

A pair of such vectors are depicted in Figure 28, showing that in general they are not colinear with the ellipsoid's center (the origin of coordinates).

For corresponding points

$$\begin{aligned} r^2 - r_0^2 &= x^2 - x_0^2 + y^2 - y_0^2 + z^2 - z_0^2 \\ &= (a^2 + \kappa - a^2) \cos^2 u_0 \cdot \sin^2 v_0 + \kappa \cdot \sin^2 u_0 \cdot \sin^2 v_0 + \kappa \cdot \cos^2 v_0 \\ &= \kappa (\sin^2 v_0 + \cos^2 v_0) = \kappa. \end{aligned} \quad (\text{A7.26})$$

That is,

$$\boxed{\kappa = r^2 - r_0^2 = \vec{r} \cdot \vec{r} - \vec{r}_0 \cdot \vec{r}_0}, \quad (\text{A7.27})$$

Corresponding points

Any two points satisfying eqs. A7.25 are called corresponding points of the confocal pair of ellipsoids. A similar relationship and its properties are valid for any two pair of confocal ellipsoids κ, κ' , but this will not be needed here.

The correspondance of points \vec{r}_0, \vec{r} may also be expressed as

$$\frac{x_0}{a} = \frac{x}{\sqrt{a^2 + \kappa}}, \quad \frac{y_0}{b} = \frac{y}{\sqrt{b^2 + \kappa}}, \quad \frac{z_0}{c} = \frac{z}{\sqrt{c^2 + \kappa}}, \quad (\text{A7.28})$$

as follows from the fact that the angular parts of each pair of components are the same. When thus expressed the correspondance provides a simple way of proving the important properties that follow.

If $\vec{r}_0, \vec{\rho}_0$ are any two points on ellipsoid A7.23 and $\vec{r}, \vec{\rho}$ its corresponding points on the confocal ellipsoid eq. A7.20, then

$$\boxed{\vec{r}_0 \cdot \vec{\rho} = \vec{r} \cdot \vec{\rho}_0.} \quad (\text{A7.29})$$

The correspondance between the components ξ, ψ, ζ of vectors $\vec{\rho}_0$ and $\vec{\rho}$ (eqs. A7.28) and eq. A7.27 give

$$\frac{\xi_0}{a} = \frac{\xi}{\sqrt{a^2 + \kappa}}, \quad \frac{\psi_0}{b} = \frac{\psi}{\sqrt{b^2 + \kappa}}, \quad \frac{\zeta_0}{c} = \frac{\zeta}{\sqrt{c^2 + \kappa}}, \quad (\text{A7.30})$$

where $\vec{\rho} \cdot \vec{\rho} - \vec{\rho}_0 \cdot \vec{\rho}_0 = \kappa$.

The proof easily follows from the expansion of the scalar product and the relationships eqs. A7.27, A7.28 and A7.30:

$$\begin{aligned} \vec{r}_0 \cdot \vec{\rho} &= x_0 \zeta + y_0 \psi + z_0 \xi = \frac{\sqrt{a^2 + \kappa}}{a} x_0 \zeta_0 + \frac{\sqrt{b^2 + \kappa}}{b} y_0 \psi_0 + \frac{\sqrt{c^2 + \kappa}}{c} z_0 \xi_0, \\ \vec{r} \cdot \vec{\rho}_0 &= x \xi_0 + y \psi_0 + z \zeta_0 = \frac{\sqrt{a^2 + \kappa}}{a} x_0 \xi_0 + \frac{\sqrt{b^2 + \kappa}}{b} y_0 \psi_0 + \frac{\sqrt{c^2 + \kappa}}{c} z_0 \zeta_0 \\ &= \vec{r}_0 \cdot \vec{\rho}. \end{aligned} \quad (\text{A7.31})$$

A corollary of eqs. A7.27 and A7.29 is

$$\boxed{|\vec{r} - \vec{\rho}_0| = |\vec{\rho} - \vec{r}_0|,} \quad (\text{A7.32})$$

as follows from

$$\begin{aligned} &(\vec{r} - \vec{\rho}_0) \cdot (\vec{r} - \vec{\rho}_0) - (\vec{\rho} - \vec{r}_0) \cdot (\vec{\rho} - \vec{r}_0) \\ &= \vec{r} \cdot \vec{r} - \vec{r} \cdot \vec{\rho}_0 - \vec{\rho}_0 \cdot \vec{r} + \vec{\rho}_0 \cdot \vec{\rho}_0 - \vec{\rho} \cdot \vec{\rho} + \vec{\rho} \cdot \vec{r}_0 + \vec{r}_0 \cdot \vec{\rho} - \vec{r}_0 \cdot \vec{r}_0 \\ &= \vec{r} \cdot \vec{r} - \vec{r}_0 \cdot \vec{r}_0 - (\vec{\rho} \cdot \vec{\rho} - \vec{\rho}_0 \cdot \vec{\rho}_0) + 2(\vec{r}_0 \cdot \vec{\rho} - \vec{r} \cdot \vec{\rho}_0) \\ &= \kappa - \kappa + 2 \cdot 0 = 0. \end{aligned} \quad (\text{A7.33})$$

This property is the foundation of Ivory's method for the calculation of the gravitational potential at points external to the body²¹⁶.

²¹⁶ MacMillan, pp. 54-57.

Appendix 8: Useful integrals

$$\int \frac{ds}{\sqrt{A+s}(B+s)^2}$$

The integrals that give the equatorial eigenvalues of both type of spheroids are of this type. In order to solve it the first step is to reduce the exponent of $A+s$ to 1 using the formula²¹⁷

$$\int \frac{ds}{\sqrt{X} \cdot Y^2} = -\frac{\sqrt{X}}{(A-B)Y} - \frac{1}{2(A-B)} \int \frac{ds}{\sqrt{X} \cdot Y}, \quad (\text{A8.1})$$

where $X = A + s, Y = B + s.$ (A8.2)

The primitives of the remaining integral are²¹⁸

$$\int \frac{ds}{\sqrt{X} \cdot Y} = \frac{2}{\sqrt{B-A}} \arctan\left(\frac{\sqrt{X}}{\sqrt{B-A}}\right) \quad \text{if } A < B, \quad (\text{A8.3})$$

$$\int \frac{ds}{\sqrt{X} \cdot Y} = \frac{1}{\sqrt{A-B}} \ln\left(\frac{\sqrt{X}-\sqrt{A-B}}{\sqrt{X}+\sqrt{A-B}}\right) \quad \text{if } A > B. \quad (\text{A8.4})$$

Therefore,

$$\int \frac{ds}{\sqrt{X} \cdot Y^2} = \frac{\sqrt{X}}{(B-A)Y} + \frac{1}{(B-A)^{3/2}} \arctan\left(\frac{\sqrt{X}}{\sqrt{B-A}}\right) \quad \text{if } A < B.$$

(A8.5)

²¹⁷ Korn & Korn, last line of p. 938 and eq. 154 of p. 941.

²¹⁸ Korn & Korn, p. 941 eq. 149.

$$\int \frac{ds}{\sqrt{X} \cdot Y^2} = -\frac{\sqrt{X}}{(A-B)Y} - \frac{1}{2} \frac{1}{(A-B)^{3/2}} \ln \left(\frac{\sqrt{X} - \sqrt{A-B}}{\sqrt{X} + \sqrt{A-B}} \right) \quad \text{if } A > B, \quad (\text{A8.6})$$

where $X = A + s, Y = B + s$.

$$\int \frac{ds}{(A+s)^{3/2} (B+s)}$$

This integral, also found for spheroids, may be reduced to the previous one when integrating by parts. Using the same notation eq. [A8.2](#),

$$\int \frac{ds}{(A+s)^{3/2} (B+s)} = \int \frac{ds}{X^{3/2} Y}. \quad (\text{A8.7})$$

Upon integration by parts

$$\int u dv = u \cdot v - \int v du, \quad \text{where} \quad (\text{A8.8})$$

$$u = \frac{1}{Y}, \quad du = -\frac{ds}{Y^2}, \quad v = -2X^{-1/2}, \quad dv = X^{-3/2},$$

it is obtained

$$\int u dv = \int \frac{ds}{X^{3/2} Y} = -\frac{2}{\sqrt{X} \cdot Y} - 2 \int \frac{ds}{\sqrt{X} \cdot Y^2}. \quad (\text{A8.9})$$

Therefore

$$\int \frac{ds}{X^{3/2} Y} = -\frac{2}{\sqrt{X} \cdot Y} - 2 \int \frac{ds}{\sqrt{X} \cdot Y^2}. \quad (\text{A8.10})$$

Upon replacement from eqs. [A8.5](#) and [A8.6](#) it is obtained

$$\int \frac{ds}{X^{3/2} Y} = -\frac{2}{\sqrt{X} \cdot Y} - \frac{2\sqrt{X}}{(B-A)Y} - \frac{2}{(B-A)^{3/2}} \arctan \left(\frac{\sqrt{X}}{\sqrt{B-A}} \right) \quad \text{if } A < B, \quad (\text{A8.11})$$

$$\int \frac{ds}{X^{3/2} Y} = \frac{2}{(A-B)\sqrt{X}} + \frac{1}{(A-B)^{3/2}} \ln \left(\frac{\sqrt{X} - \sqrt{A-B}}{\sqrt{X} + \sqrt{A-B}} \right) \quad \text{if } A > B. \quad (\text{A8.12})$$

Appendix 9: Legendre's elliptic integrals

Definitions

$F(\phi, k)$ and $E(\phi, k)$ are Legendre's incomplete elliptic integrals²¹⁹ of the first and second kind, respectively, considered to be a generalization of the well known trigonometric functions. Though not often mentioned in the theory of electromagnetism, they appear in common problems such as the computation of the arc length of an ellipse (from which they got their name), the dependence of the period of a pendulum on the amplitude of oscillation²²⁰ and the surface area of an ellipsoid (see eq. A7.5).

The canonical expressions for the elliptic integrals are²²¹:

$$\begin{aligned} E(\phi, k) &\equiv \int_0^\phi \sqrt{1 - k^2 \sin^2 t} \, dt \equiv \int_0^\phi \sqrt{1 - \sin^2 \alpha \cdot \sin^2 t} \, dt = E(\phi \setminus \alpha), \\ F(\phi, k) &\equiv \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \equiv \int_0^\phi \frac{dt}{\sqrt{1 - \sin^2 \alpha \cdot \sin^2 t}} = F(\phi \setminus \alpha) \end{aligned} \quad (\text{A9.1})$$

ϕ is the amplitude, k the elliptic modulus and α the modular angle of these integrals. Their values can be found in double entry tables like those of Abramowitz and Stegun (pp. 613-618), and may be calculated for arbitrary values of their two arguments by mathematical software like Mathematica™ and Maple™.

Notation used in the more important references

Different authors use different conventions for the arguments, both in order and meaning. The following table shows the relationship between the notation used in this book and that of the authors more often quoted here. Before making any calculations the reader should carefully check those used by its favourite or available author or software, for which it is recommended to check one or more values, as those given in [Problem 21](#).

²¹⁹ Complete elliptic integrals have $\phi = \pi/2$.

²²⁰ See, for instance. C. E. Soliv  rez, [Comportamiento de un p  ndulo real sin rozamiento](#).

²²¹ Korn & Korn, p. 833, eqs. 21.6-29a; p. 834, eqs. 21.6-29c and 21.6-29d. Weisstein, Eric W; [Elliptic Integral of the First Kind](#); MathWorld--A Wolfram Web Resource. Abramowitz & Stegun, pp. 589 eqs. 17.2.8 and 17.2.6.

Author	N_x, N_y, N_z	β	γ	k	$E(\phi, k), F(\phi, k)$	pp.
Abramowitz & Stegun	— ²²²	—	—	$\sin \alpha$	$E(\phi \backslash \alpha), F(\phi \backslash \alpha);$ $E(\phi m), F(\phi m)$ ²²³	589-590
Korn & Korn	— ²²⁴	—	—	$\sin \alpha$	$E(k, \phi), F(k, \phi);$ $E(\phi \backslash \alpha), F(\phi \backslash \alpha)$	833-836
MacMillan	— ²²⁵	β	γ	k	$E(\omega, k), F(\omega, k)$	58-60
Osborn	$L/4\pi,$ $M/4\pi,$ $N/4\pi$	$\cos \vartheta$	$\cos \phi$	$\sin \alpha$	$E(k, \vartheta), F(k, \vartheta)$	351-352
Stoner	D_a, D_b, D_c	β	γ	k	$E(k, \phi), F(k, \phi)$	807-808
Mathematica	—	—	—	k	EllipticE[ϕ, k^2], EllipticF[ϕ, k^2],	

Table 6. Notation used by several authors.

Although all arguments are here real, for a full analysis of the general properties of elliptic integrals they should be taken to be complex numbers. In respect to their interest in this book —the expressions for the components of the depolarization tensor of the triaxial ellipsoid, eqs. 4.9 and — their arguments are real, bounded and given by ratios of the ellipsoid’s semiaxes, as follows:

$$0 \leq \sin \phi = \sqrt{1 - \gamma^2} \leq 1, \quad 0 \leq k = \sqrt{\frac{1 - \beta^2}{1 - \gamma^2}} = \sin \alpha \leq 1, \quad (\text{A9.2})$$

where $\beta = \frac{b}{a}, \gamma = \frac{c}{a}.$

Special values and parametric graphs

Some special values of the elliptic functions are given in the following table for its use in the calculation of depolarization tensors. In the case of the constant interior tensor \mathbf{N} , the values of the arguments are connected in a one to one relationship with those of β and γ , as given in the expression of the auxiliary function f for the triaxial ellipsoid eq. 4.52. These values are also shown in the table, identifying the type of ellipsoid they characterize.

²²² Gives general properties and tables of values.

²²³ $m = \sin^2 \alpha.$

²²⁴ Gives general properties and graphs.

²²⁵ Gives only the expression of the potential.

ϕ	k	α	β	γ	Ellipsoid's type	$E(\phi, k)$	$F(\phi, k)$
0	k	α	1	1	sphere	0 ²²⁶	0 ²²⁷
ϕ	0	0	1	γ	oblate spheroid	ϕ ²²⁸	ϕ ²²⁹
ϕ	1	$\pi/2$	β	β	prolate spheroid	$\sin \phi$ ₂₃₀	$\ln(\tan(\phi/2 + \pi/4))$ ²³¹
$\pi/2$	1	$\pi/2$	0	0	infinite elliptic cylinder ²³²	1	∞
$\pi/2$	0	0	1 ²³³	0	infinite sheet of constant thickness	$\pi/2$	$\pi/2$ ²³⁴

Table 7. Special values of the incomplete elliptic functions $E(\phi, k)$ and $F(\phi, k)$.

Figure 29 and Figure 30 show parametric graphs of $E(\phi \backslash \alpha)$ and $F(\phi \backslash \alpha)$. In both cases the left side graph shows the dependence on α for ϕ taken as a parameter (there written φ). The right side shows the dependence on ϕ , where each curve corresponds to a different value of parameter α .

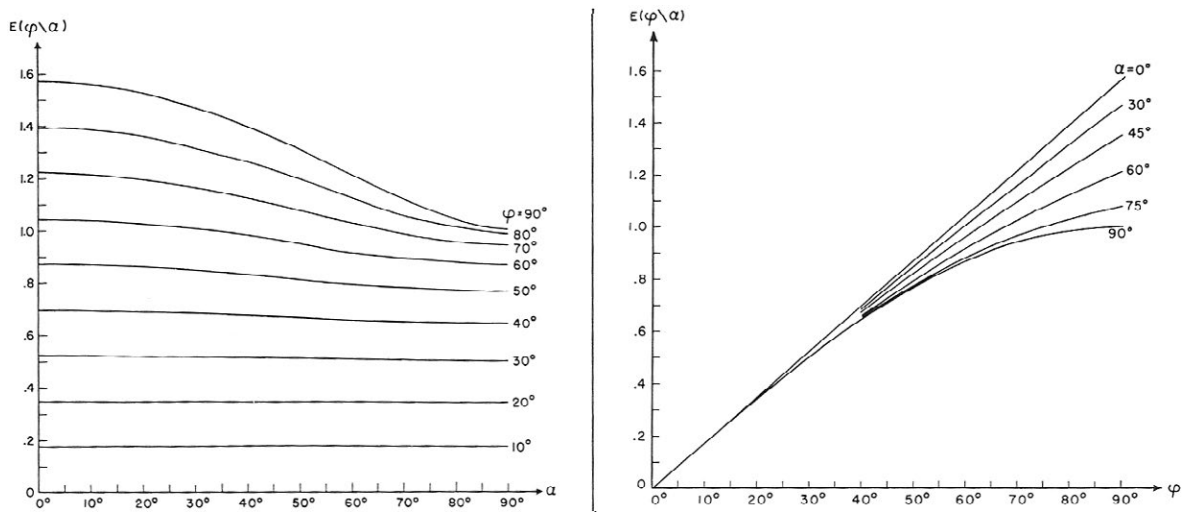


Figure 29. Parametric graph of the incomplete elliptic function of the second kind E .²³⁵

²²⁶ Eq. A9.1.

²²⁷ Eq. A9.1.

²²⁸ Abramowitz & Stegun, eq. 17.4.23.

²²⁹ Abramowitz & Stegun, eq. 17.4.19.

²³⁰ Abramowitz & Stegun, eq. 17.4.25.

²³¹ Abramowitz & Stegun, eq. 17.4.21.

²³² MacMillan, p. 70.

²³³ See the comments on the infinite sheet in section *Infinite semiaxes*.

²³⁴ Abramowitz & Stegun, Fig. 17.2.

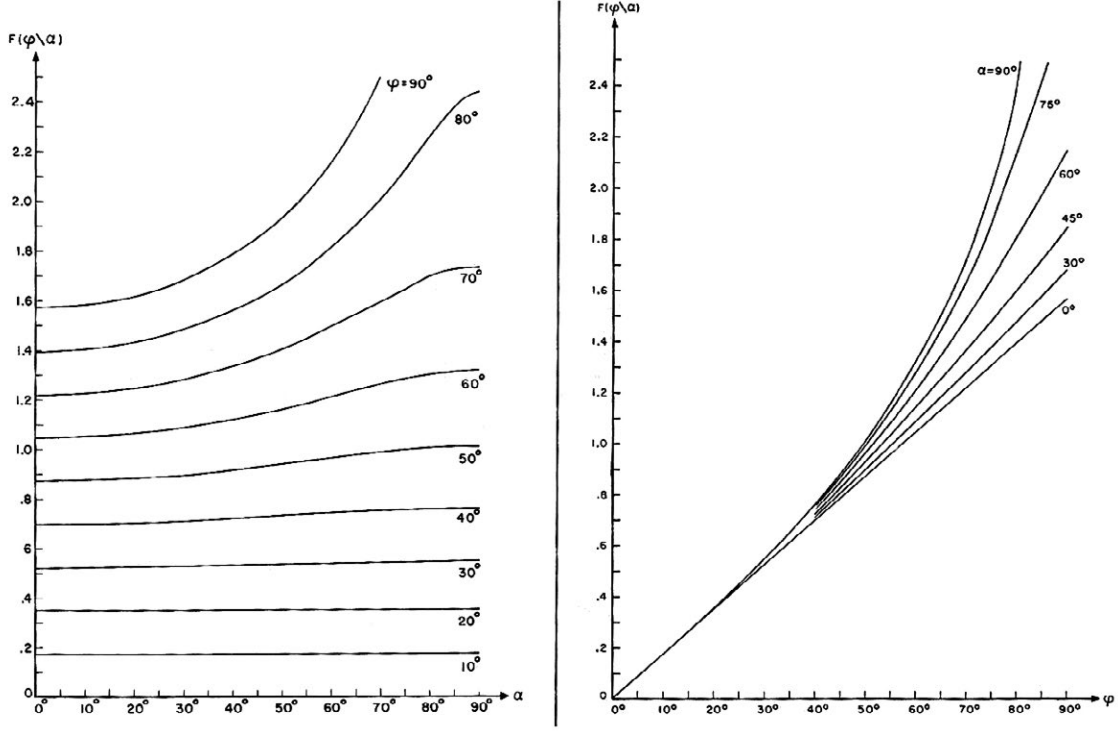


Figure 30. Parametric graph of the incomplete elliptic function of the first kind $F(\phi|\alpha)$.²³⁶

As seen in Figure 29, $E(\phi|\alpha)$ is never negative. Its highest value in the intervals determined by eqs. A9.2 is $E(\pi/2|0) = \pi/2$ ²³⁷. Its lowest value is attained at the origin, where $E(0|0) = 0$. For constant ϕ , $E(\phi|\alpha)$ is a decreasing function of α which is almost constant for the lowest values of ϕ . For constant α it is an increasing function of ϕ , linear for $\alpha = 0$ (the line $E(\phi|0) = \phi$ in Table 7).

As seen in Figure 30, $F(\phi|\alpha)$ is never negative and diverges at $F(\pi/2|\pi/2)$. For fixed α it is almost constant for low values ϕ (ϕ in the figure). For $\alpha = 0$, is linear in ϕ (the line $E(\phi|0) = \phi$ in Table 7).

Reduction to normal form of the elliptic integrals of interest

The integrals of interest for the calculation of the depolarization tensor are²³⁸

$$\text{For } a \geq b \geq c, k = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \sin \phi = \sqrt{\frac{a^2 - c^2}{a^2 + \kappa}}, \frac{x^2}{a^2 + \kappa} + \frac{y^2}{b^2 + \kappa} + \frac{z^2}{c^2 + \kappa} = 1, \quad (\text{A9.3})$$

²³⁵ Abramowitz & Stegun, p. 594, figures 17.6 and 17.7.

²³⁶ Abramowitz & Stegun, p. 594, figures 17.3 and 17.4.

²³⁷ This corresponds to the complete elliptic integral of the second kind. Its graph is given in Abramowitz & Stegun, p. 592 Figure 17.2.

²³⁸ MacMillan, pp. 58-60, where the expression for function dn in p. 60 should be $\text{dn } v_\kappa = \sqrt{\frac{b^2 + \kappa}{a^2 + \kappa}}$,

as follows from its definition at p 58, and $E(\omega_\kappa)$ is $E(\omega_\kappa, k)$.

$$\int_{\kappa}^{\infty} \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = \frac{2}{\sqrt{a^2-c^2}} F(\phi, k), \quad (\text{A9.4})$$

$$\int_{\kappa}^{\infty} \frac{ds}{(a^2+s)^{3/2} \sqrt{(b^2+s)(c^2+s)}} = \frac{2}{\sqrt{a^2-c^2}(a^2-b^2)} [-E(\phi, k) + F(\phi, k)], \quad (\text{A9.5})$$

$$\begin{aligned} \int_{\kappa}^{\infty} \frac{ds}{(b^2+s)^{3/2} \sqrt{(a^2+s)(c^2+s)}} = & -\frac{2\sqrt{c^2+\kappa}}{\sqrt{(a^2+\kappa)(b^2+\kappa)(b^2-c^2)}} \\ & + \frac{2\sqrt{a^2-c^2}}{(a^2-b^2)(b^2-c^2)} E(\phi, k) - \frac{2}{\sqrt{a^2-c^2}(a^2-b^2)} F(\phi, k), \end{aligned} \quad (\text{A9.6})$$

$$\begin{aligned} & \int_{\kappa}^{\infty} \frac{ds}{(c^2+s)^{3/2} \sqrt{(a^2+s)(b^2+s)}} \\ = & \frac{2\sqrt{b^2+\kappa}}{\sqrt{(a^2+\kappa)(c^2+\kappa)(b^2-c^2)}} - \frac{2}{\sqrt{a^2-c^2}(b^2-c^2)} E(\phi, k). \end{aligned} \quad (\text{A9.7})$$

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ISBN 978-987-28304-0-3



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